Implementation of a Large-Scale Optimal Power Flow Solver Based on Semidefinite Programming

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***Abstract*—The application of semidefinite programming to the optimal power flow (OPF) problem has recently attracted significant research interest. This paper provides advances in modeling and computation required for solving the OPF problem for large-scale, general power system models. Specifically, a semidefinite programming relaxation of the OPF problem is presented that incorporates multiple generators at the same bus and parallel lines. Recent research in matrix completion techniques that decompose a single large matrix constrained to be positive semidefinite into many smaller matrices has made solution of OPF problems using semidefinite programming computationally tractable for large system models. We provide three advances to existing decomposition techniques: a matrix combination algorithm that further decreases solver time, a modification to an existing decomposition technique that extends its applicability to general power system networks, and a method for obtaining the optimal voltage profile from the solution to a decomposed semidefinite program.**

***Index Terms*—Optimal power flow, Semidefinite optimization**

I. Introduction

THE optimal power flow (OPF) problem determines an optimal operating point for an electric power system in terms of a specified objective function, subject to both network equality constraints (i.e., the power flow equations, which model the relationship between voltages and power injections) and engineering limits (e.g., inequality constraints on voltage magnitudes, active and reactive power generations, and flows on transmission lines and transformers). Total variable generation cost per unit time is typically chosen as the objective function.

The OPF problem is typically nonconvex due to the nonlinear power flow equations [1]. Nonconvexity of the OPF problem has made solution techniques an ongoing research topic. Many OPF solution techniques have been proposed, including successive quadratic programs, Lagrangian relaxation, genetic algorithms, particle swarm optimization, and interior point methods [2], [3].

Recently, significant attention has focused on the application of semidefinite programming to the OPF problem [4], [5]. Using a rank relaxation, the OPF problem is reformulated as a convex semidefinite program. If the relaxed problem satisfies a rank condition (i.e., the semidefinite program has zero duality gap), the globally optimal solution to the original OPF problem can be determined in polynomial time. No prior OPF solution method offers a guarantee of finding a global solution in polynomial time; semidefinite programming approaches thus have a substantial advantage over traditional solution techniques. Note, however, that the rank condition is not always satisfied, which means that semidefinite relaxations do not give physically meaningful solutions for all realistic power system models [6], [7]. The solution to the semidefinite relaxation for such cases may provide a good starting point for a local search algorithm. Further, the solution to the semidefinite relaxation always yields a lower bound to the unknown global solution, which provides a measure of sub-optimality for any local solution. It is important to note, however, that a solution to the semidefinite relaxation with non-zero duality gap is not necessarily a better approximation to the true solution than other solution techniques. For instance, when applied to the three-bus system in [6], the DC OPF, a common linear approximation to the OPF problem [3], yields more accurate active power generation and active power Lagrange multipliers (LMPs), as compared to a traditional AC OPF calculation, than the solution with non-zero duality gap from the semidefinite relaxation.

Recent research has investigated conditions under which the rank condition is satisfied; to date, sufficient conditions for rank condition satisfaction include highly limiting requirements on power injection and voltage magnitude limits and either radial networks (typical of distribution system models) or unrealistically dense placement of controllable phase shifting transformers [8]–[11]. Additional work includes using semidefinite programming to create voltage stability margins [12].

This paper first focuses on modeling aspects that must be addressed in order to apply the semidefinite program to general power system models. The first issue addressed is that of allowing multiple generators at the same bus. By equating bus power injections with power generation, existing formulations only allow a single generator to exist at a bus. We use the concept of equal marginal generation cost to produce a formulation allowing for multiple generators at the same bus, each with separate cost functions and generation limits. This paper considers both quadratic and convex piecewise-linear generator cost functions.

A method for incorporating flow limits on parallel lines is then presented. Existing formulations limit the total flow between two buses, which cannot properly account for parallel lines with different electrical properties and flow limits. In contrast, the proposed method limits the flow on each individual line. Lines in this formulation can have off-nominal voltage ratios and/or non-zero phase shifts.

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This paper next advances research in the computational tractability of applying semidefinite programming to large power system models. Semidefinite programming relaxations of the OPF problem constrain a $2n \times 2n$ symmetric matrix to be positive semidefinite, where n is the number buses in the system. The semidefinite program size thus grows as the square of the number of buses, which makes solution of the OPF problem by semidefinite programming computationally challenging for large systems. Recent work using matrix completion [13]–[15] reduces the computational burden inherent in solving large systems by taking advantage of the sparse matrix structure created by realistic power system models. Sojoudi and Lavaei [10], Jabr [16], and Bai and Wei [17] present formulations that decompose the single large $2n \times 2n$ positive semidefinite matrix constraint into positive semidefinite constraints on many smaller matrices. If the matrices from these decompositions satisfy a rank condition, the $2n \times 2n$ matrix also satisfies the rank condition and the optimal solution can be obtained. Sojoudi and Lavaei's decomposition [10] uses a cycle basis of the network. The matrix decomposition approach used by both Jabr [16] and Bai and Wei [17] is based on the maximal cliques of a chordal extension of the network.

We provide several enhancements to the existing decompositions. Specifically, we present a heuristic algorithm for combining some of the small matrices resulting from the decomposition. Since linking constraints are required between elements of the decomposed matrices that refer to the same element in the $2n \times 2n$ matrix, it is not always advantageous to create the smallest possible matrices. Combining matrices eliminates some of these linking constraints, which can result in significant computational speed increases. We justify the claim that the proposed algorithm can substantially increase computational speed using both theoretical arguments and several test cases.

Note that this paper considers a centralized application of these decompositions in the sense of creating one semidefinite program that is solved on a single computer as opposed to creating many subproblems that are solved using decentralized techniques as in [18]. Centralized application allows for solution with existing generic semidefinite programming solvers.

A further enhancement presented in this paper is a technique for recovering the optimal voltage profile from the decomposed matrices. While the steps are relatively straightforward, existing literature does not detail a method for actually obtaining the optimal voltage profile from a solution to a decomposed formulation.

Although we focus on the maximal clique decomposition proposed by both Jabr [16] and Bai and Wei [17] due to the voluminous literature on matrix completion with chordal extensions (e.g., [13]–[15]), both of these enhancements could be applied to Sojoudi and Lavaei's decomposition [10] as well.

We also describe a modification to the maximal clique decomposition as formulated by Jabr [16] that allows for application to general power systems. This formulation creates a chordal extension of the network using a Cholesky factorization of the absolute value of the imaginary part of the bus admittance matrix. However, this matrix may fail to be positive definite (for instance, in networks with significant shunt capacitive compensation), thus preventing calculation of a Cholesky factorization. We describe an alternative matrix that is always positive definite and gives an equivalent chordal extension, thus broadening the applicability of this decomposition to general networks.

This paper is organized as follows. Section II provides both the classical formulation of the OPF problem and a proposed semidefinite programming relaxation that incorporates multiple generators at the same bus and parallel lines, including lines with off-nominal voltage ratios and/or non-zero phase-shifts. Section III first gives an overview of the maximal clique decomposition and then presents three advances in decompositions for large-scale system models: an algorithm that improves computation speed by combining matrices, a modification to Jabr's maximal clique decomposition that extends its applicability to general power system networks, and a technique for recovering the optimal voltage profile from a solution to a decomposed formulation. Section III also discusses rank condition satisfaction for large system models.

II. The OPF Problem and Modeling Issues

We first present the OPF problem as it is classically formulated. Specifically, this formulation is in terms of rectangular voltage coordinates, active and reactive power generation, and apparent power line-flow limits. Each bus may have multiple generators, and parallel lines are allowed. This classical OPF formulation is generally nonconvex. We then describe a semidefinite programming relaxation of the OPF problem adopted from [5] that handles the modeling issues of multiple generators at the same bus and parallel lines.

A. Classical OPF Formulation

Consider an n-bus power system, where $\mathcal{N} = \{1, 2, \ldots, n\}$ represents the set of all buses. Define $\mathcal{G}$ as the set of all generators, with $\mathcal{G}_i$ the set of generators at bus i. Let $\mathcal{G}^q$ represent the set of all generators with quadratic cost functions, with $\mathcal{G}_i^q$ those such generators at bus i. Let $\mathcal{G}^{pw}$ represent the set of generators with piecewise-linear cost functions, with $\mathcal{G}_i^{pw}$ those such generators at bus i. (Some of these sets may be empty.) $P_{Gg} + jQ_{Gg}$ represents the active and reactive power output of generator $g \in \mathcal{G}$. $P_{Di} + jQ_{Di}$ represents the active and reactive load demand at each bus $i \in \mathcal{N}$. $V_i = V_{di} + jV_{qi}$ represents the voltage phasors in rectangular coordinates at each bus $i \in \mathcal{N}$. Superscripts "max" and "min" denote specified upper and lower limits. Let $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ denote the network admittance matrix.

$\mathcal{L}$ represents the set of all lines, where line $k \in \mathcal{L}$ has terminals at buses l_k and m_k, with parallel lines allowed (i.e., more than one line between the same terminal buses). Let S_k represent the apparent power flow on the line $k \in \mathcal{L}$.

Define a cost function associated with each generator, typically representing a dollar/hour variable operating cost. This paper considers quadratic and convex piecewise-linear cost functions in (1a) and (1b), respectively. In (1a), the terms c_{g2}, c_{g1}, and c_{g0} represent the quadratic cost coefficients for generator $g \in \mathcal{G}^q$. In (1b), generator $g \in \mathcal{G}^{pw}$ has a piecewise-linear cost function composed of r_g line segments specified

by slopes $m_{g1}, \ldots, m_{gr_g}$ and breakpoints (a_{gt}, b_{gt}), $t = 1, \ldots, r_g$, where a_{gt} is the power generation coordinate and b_{gt} is the cost coordinate for the breakpoint.

$$\mathcal{C}_g^q(P_{Gg}) = c_{g2}P_{Gg}^2 + c_{g1}P_{Gg} + c_{g0} \tag{1a}$$

$$\mathcal{C}_g^{pw}(P_{Gg}) = \begin{cases} m_{g1}(P_{Gg} - a_{g1}) + b_{g1}, & P_{Gg} \le a_{g1} \\ m_{g2}(P_{Gg} - a_{g2}) + b_{g2}, & a_{g1} < P_{Gg} \le a_{g2} \\ \vdots & \vdots \\ m_{gr}(P_{Gg} - a_{gr}) + b_{gr}, & a_{gr} < P_{Gg} \end{cases} \tag{1b}$$

The classical OPF problem is then

$$\min \sum_{g\in\mathcal{G}^q} \mathcal{C}_g^q(P_{Gg}) + \sum_{g\in\mathcal{G}^{pw}} \mathcal{C}_g^{pw}(P_{Gg}) \tag{2a}$$

subject to

$$P_{Gg}^{\min} \le P_{Gg} \le P_{Gg}^{\max} \qquad \forall g \in \mathcal{G} \tag{2b}$$

$$Q_{Gg}^{\min} \le Q_{Gg} \le Q_{Gg}^{\max} \qquad \forall g \in \mathcal{G} \tag{2c}$$

$$\left(V_i^{\min}\right)^2 \le V_{di}^2 + V_{qi}^2 \le \left(V_i^{\max}\right)^2 \qquad \forall i \in \mathcal{N} \tag{2d}$$

$$|S_k| \le S_k^{\max} \qquad \forall k \in \mathcal{L} \tag{2e}$$

$$\sum_{g\in\mathcal{G}_i}(P_{Gg}) - P_{Di} = V_{di}\sum_{h=1}^{n}(\mathbf{G}_{ih}V_{dh} - \mathbf{B}_{ih}V_{qh}) + V_{qi}\sum_{h=1}^{n}(\mathbf{B}_{ih}V_{dh} + \mathbf{G}_{ih}V_{qh}) \qquad \forall i \in \mathcal{N} \tag{2f}$$

$$\sum_{g\in\mathcal{G}_i}(Q_{Gg}) - Q_{Di} = V_{di}\sum_{h=1}^{n}(-\mathbf{B}_{ih}V_{dh} - \mathbf{G}_{ih}V_{qh}) + V_{qi}\sum_{h=1}^{n}(\mathbf{G}_{ih}V_{dh} - \mathbf{B}_{ih}V_{qh}) \qquad \forall i \in \mathcal{N} \tag{2g}$$

Note that this formulation limits the apparent power flow measured at each end of a given line, recognizing that active and reactive line losses can cause these quantities to differ.

B. Semidefinite Programming Relaxation of the OPF Problem

This section first describes the semidefinite relaxation of the OPF problem, including the capability to incorporate parallel lines and multiple generators at the same bus. Let e_i denote the i^{th} standard basis vector in $\mathbb{R}^n$. Define the matrix $Y_i = e_i e_i^T \mathbf{Y}$, where the superscript T indicates the transpose operator.

Matrices employed in the bus power injection and voltage magnitude constraints are

$$\mathbf{Y}_i = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\left(Y_i + Y_i^T\right) & \mathrm{Im}\left(Y_i^T - Y_i\right) \\ \mathrm{Im}\left(Y_i - Y_i^T\right) & \mathrm{Re}\left(Y_i + Y_i^T\right) \end{bmatrix} \tag{3}$$

$$\bar{\mathbf{Y}}_i = -\frac{1}{2}\begin{bmatrix} \mathrm{Im}\left(Y_i + Y_i^T\right) & \mathrm{Re}\left(Y_i - Y_i^T\right) \\ \mathrm{Re}\left(Y_i^T - Y_i\right) & \mathrm{Im}\left(Y_i + Y_i^T\right) \end{bmatrix} \tag{4}$$

$$\mathbf{M}_i = \begin{bmatrix} e_i e_i^T & \mathbf{0} \\ \mathbf{0} & e_i e_i^T \end{bmatrix} \tag{5}$$

A "line" in this formulation includes both transmission lines and transformers, where transformers may have both a phase shift and/or an off-nominal voltage ratio. That is, line k is modeled as a Π circuit (with series admittance $g_k + jb_k$ and shunt capacitances $\frac{b_{sh,k}}{2}$) in series with an ideal transformer (with turns ratio $1 : \tau_k e^{j\theta_k}$) as in [3]. Note that a small minimum resistance is enforced on all lines in accordance with [5]. Define f_i as the i^{th} standard basis vector in $\mathbb{R}^{2n}$. Matrices employed in the line-flow constraints are

$$\begin{aligned}\mathbf{Z}_{k_l} = {} & \frac{g_k}{\tau_k^2}\left(f_{l_k}f_{l_k}^T + f_{l_k+n}f_{l_k+n}^T\right) \\ & - c_l\left(f_{l_k}f_{m_k}^T + f_{m_k}f_{l_k}^T + f_{l_k+n}f_{m_k+n}^T + f_{m_k+n}f_{l_k+n}^T\right) \\ & + s_l\left(f_{l_k}f_{m_k+n}^T + f_{m_k+n}f_{l_k}^T - f_{l_k+n}f_{m_k}^T - f_{m_k}f_{l_k+n}^T\right)\end{aligned} \tag{6}$$

$$\begin{aligned}\mathbf{Z}_{k_m} = {} & g_k\left(f_{m_k}f_{m_k}^T + f_{m_k+n}f_{m_k+n}^T\right) \\ & - c_m\left(f_{l_k}f_{m_k}^T + f_{m_k}f_{l_k}^T + f_{l_k+n}f_{m_k+n}^T + f_{m_k+n}f_{l_k+n}^T\right) \\ & + s_m\left(f_{l_k+n}f_{m_k}^T + f_{m_k}f_{l_k+n}^T - f_{l_k}f_{m_k+n}^T - f_{m_k+n}f_{l_k}^T\right)\end{aligned} \tag{7}$$

$$\begin{aligned}\bar{\mathbf{Z}}_{k_l} = {} & -\left(\frac{2b_k + b_{sh,k}}{2\tau_k^2}\right)\left(f_{l_k}f_{l_k}^T + f_{l_k+n}f_{l_k+n}^T\right) \\ & + c_l\left(f_{l_k}f_{m_k+n}^T + f_{m_k+n}f_{l_k}^T - f_{l_k+n}f_{m_k}^T - f_{m_k}f_{l_k+n}^T\right) \\ & + s_l\left(f_{l_k}f_{m_k}^T + f_{m_k}f_{l_k}^T + f_{l_k+n}f_{m_k+n}^T + f_{m_k+n}f_{l_k+n}^T\right)\end{aligned} \tag{8}$$

$$\begin{aligned}\bar{\mathbf{Z}}_{k_m} = {} & -\left(b_k + \frac{b_{sh,k}}{2}\right)\left(f_{m_k}f_{m_k}^T + f_{m_k+n}f_{m_k+n}^T\right) \\ & + c_m\left(f_{l_k+n}f_{m_k}^T + f_{m_k}f_{l_k+n}^T - f_{l_k}f_{m_k+n}^T - f_{m_k+n}f_{l_k}^T\right) \\ & + s_m\left(f_{l_k}f_{m_k}^T + f_{m_k}f_{l_k}^T + f_{l_k+n}f_{m_k+n}^T + f_{m_k+n}f_{l_k+n}^T\right)\end{aligned} \tag{9}$$

where, for notational convenience,

$$c_l = \left(g_k\cos(\theta_k) + b_k\cos\left(\theta_k + \frac{\pi}{2}\right)\right)/(2\tau_k) \tag{10}$$

$$c_m = \left(g_k\cos(-\theta_k) + b_k\cos\left(-\theta_k + \frac{\pi}{2}\right)\right)/(2\tau_k) \tag{11}$$

$$s_l = \left(g_k\sin(\theta_k) + b_k\sin\left(\theta_k + \frac{\pi}{2}\right)\right)/(2\tau_k) \tag{12}$$

$$s_m = \left(g_k\sin(-\theta_k) + b_k\sin\left(-\theta_k + \frac{\pi}{2}\right)\right)/(2\tau_k) \tag{13}$$

To write the semidefinite relaxation, first define the vector of voltage coordinates

$$x = \begin{bmatrix} V_{d1} & V_{d2} & \ldots & V_{dn} & V_{q1} & V_{q2} & \ldots & V_{qn} \end{bmatrix} \tag{14}$$

Then define the rank one matrix

$$\mathbf{W} = xx^T \tag{15}$$

The active and reactive power injections at bus i are then given by $\mathrm{tr}(\mathbf{Y}_i\mathbf{W})$ and $\mathrm{tr}\left(\bar{\mathbf{Y}}_i\mathbf{W}\right)$, respectively, where tr indicates the matrix trace operator (i.e., sum of the diagonal elements). The square of the voltage magnitude at bus i is given by $\mathrm{tr}(\mathbf{M}_i\mathbf{W})$.

Similarly, the active and reactive line flows for line $k \in \mathcal{L}$ at terminal bus l are given by $\mathrm{tr}(\mathbf{Z}_{k_l}\mathbf{W})$ and $\mathrm{tr}\left(\bar{\mathbf{Z}}_{k_l}\mathbf{W}\right)$. Due to the asymmetry introduced by allowing transformers with off-nominal voltage ratios and non-zero phase shifts, we also

require separate matrices to represent active and reactive power flows from the other terminal of line k at bus m: $\mathrm{tr}\left(\mathbf{Z}_{k_m}\mathbf{W}\right)$ and $\mathrm{tr}\left(\bar{\mathbf{Z}}_{k_m}\mathbf{W}\right)$, respectively.

Replacement of the rank one constraint (15) by the less stringent constraint $\mathbf{W} \succeq 0$, where $\succeq 0$ indicates the corresponding matrix is positive semidefinite, yields the semidefinite relaxation. The semidefinite relaxation is "tight" when the solution has zero duality gap. The duality gap for a solution to the semidefinite relaxation refers to the difference between the optimal objective value for the semidefinite relaxation and the objective value for a global solution to the classical formulation of the OPF problem (2). A solution to the semidefinite relaxation has zero duality gap if and only if the rank condition $\mathrm{rank}\left(\mathbf{W}\right) \leq 2$ is satisfied. For a solution with zero duality gap, a unique rank one matrix $\mathbf{W}$ can be recovered by enforcing the known voltage angle at the reference bus [5].

The semidefinite relaxation of the OPF problem is

$$\min \sum_{g\in\mathcal{G}^q} \alpha_g + \sum_{g\in\mathcal{G}^{pw}} \beta_g \tag{16a}$$

subject to

$$P_{Gg}^{\min} \leq P_{Gg} \leq P_{Gg}^{\max} \qquad \forall g \in \mathcal{G} \tag{16b}$$

$$-P_{Di} + \sum_{g\in\mathcal{G}_i} P_{Gg} = \mathrm{tr}\left(\mathbf{Y}_i\mathbf{W}\right) \qquad \forall i \in \mathcal{N} \tag{16c}$$

$$Q_i^{\min} \leq \mathrm{tr}\left(\bar{\mathbf{Y}}_i\mathbf{W}\right) \leq Q_i^{\max} \qquad \forall i \in \mathcal{N} \tag{16d}$$

$$\left(V_i^{\min}\right)^2 \leq \mathrm{tr}\left(\mathbf{M}_i\mathbf{W}\right) \leq \left(V_i^{\max}\right)^2 \qquad \forall i \in \mathcal{N} \tag{16e}$$

$$\begin{bmatrix} -\left(S_k^{\max}\right)^2 & \mathrm{tr}\left(\mathbf{Z}_{k_l}\mathbf{W}\right) & \mathrm{tr}\left(\bar{\mathbf{Z}}_{k_l}\mathbf{W}\right) \\ \mathrm{tr}\left(\mathbf{Z}_{k_l}\mathbf{W}\right) & -1 & 0 \\ \mathrm{tr}\left(\bar{\mathbf{Z}}_{k_l}\mathbf{W}\right) & 0 & -1 \end{bmatrix} \preceq 0 \qquad \forall k \in \mathcal{L} \tag{16f}$$

$$\begin{bmatrix} -\left(S_k^{\max}\right)^2 & \mathrm{tr}\left(\mathbf{Z}_{k_m}\mathbf{W}\right) & \mathrm{tr}\left(\bar{\mathbf{Z}}_{k_m}\mathbf{W}\right) \\ \mathrm{tr}\left(\mathbf{Z}_{k_m}\mathbf{W}\right) & -1 & 0 \\ \mathrm{tr}\left(\bar{\mathbf{Z}}_{k_m}\mathbf{W}\right) & 0 & -1 \end{bmatrix} \preceq 0 \qquad \forall k \in \mathcal{L} \tag{16g}$$

$$\begin{bmatrix} c_{g1}P_{Gg} + c_{g0} - \alpha_g & \sqrt{c_{g2}}P_{Gg} \\ \sqrt{c_{g2}}P_{Gg} & -1 \end{bmatrix} \preceq 0 \qquad \forall g \in \mathcal{G}^q \tag{16h}$$

$$\left\{\beta_g \geq m_{gt}\left(P_{Gg} - a_{gt}\right) + b_{gt} \quad \forall t = 1,\ldots,r_g\right\} \qquad \forall g \in \mathcal{G}^{pw} \tag{16i}$$

$$\mathbf{W} \succeq 0 \tag{16j}$$

where apparent power line-flow limits and quadratic generator cost functions are implemented using Schur's complement formula in (16f)–(16g) and (16h), respectively; in (16i), the piecewise-linear generator cost functions are implemented using the "constrained cost variable" method as in [3]; and, for notational convenience, the maximum and minimum reactive power injections at each bus are defined as

$$Q_i^{\max} = -Q_{Di} + \sum_{g\in\mathcal{G}_i} Q_{Gg}^{\max} \tag{17}$$

$$Q_i^{\min} = -Q_{Di} + \sum_{g\in\mathcal{G}_i} Q_{Gg}^{\min} \tag{18}$$

The dual form of the semidefinite relaxation (i.e., the dual of (16)) requires definition of Lagrange multipliers corresponding to each constraint in (16). Define vectors of Lagrange multipliers associated with lower inequality bounds on active power, reactive power, and voltage magnitude as $\underline{\psi}_k$, $\underline{\gamma}_i$, and $\underline{\mu}_i$, and those associated with upper bounds as $\bar{\psi}_k$, $\bar{\gamma}_i$, and $\bar{\mu}_i$, respectively. Define a scalar variable λ_i as the aggregated Lagrange multiplier (i.e., the locational marginal price (LMP)) of active power at each bus i. Note that λ_i is not constrained to be non-negative. Define two 3×3 symmetric matrices per line for the line-flow limits measured from each line terminal: $\mathbf{H}_{k_l}$ and $\mathbf{H}_{k_m}$, with $\mathbf{H}_{k_l}^{cd}$ and $\mathbf{H}_{k_m}^{cd}$ indicating the (c,d) element of the corresponding matrix. Define 2×2 symmetric matrices for each generator with a quadratic cost function: $\mathbf{R}_g$, with $\mathbf{R}_g^{cd}$ the (c,d) element of $\mathbf{R}_g$. Define a Lagrange multiplier ζ_{gt} for each line segment t of each generator g with a piecewise-linear cost function.

Define a matrix-valued function $\mathbf{A}$.

$$\begin{aligned}\mathbf{A} = &\sum_{i\in\mathcal{N}} \left\{\lambda_i\mathbf{Y}_i + \left(\bar{\gamma}_i - \underline{\gamma}_i\right)\bar{\mathbf{Y}}_i + \left(\bar{\mu}_i - \underline{\mu}_i\right)\mathbf{M}_i\right\} \\ &+ 2\sum_{k\in\mathcal{L}} \left\{\mathbf{H}_{k_l}^{12}\mathbf{Z}_{k_l} + \mathbf{H}_{k_m}^{12}\mathbf{Z}_{k_m} + \mathbf{H}_{k_l}^{13}\bar{\mathbf{Z}}_{k_l} + \mathbf{H}_{k_m}^{13}\bar{\mathbf{Z}}_{k_m}\right\}\end{aligned} \tag{19}$$

Define a scalar real-valued function ρ.

$$\begin{aligned}\rho = &\sum_{i\in\mathcal{N}} \Bigg\{\lambda_i P_{Di} + \underline{\gamma}_i Q_i^{\min} - \bar{\gamma}_i Q_i^{\max} + \underline{\mu}_i\left(V_i^{\min}\right)^2 - \bar{\mu}_i\left(V_i^{\max}\right)^2 \\ &+ \sum_{g\in\mathcal{G}_i^q}\left(\underline{\psi}_g P_{Gg}^{\min} - \bar{\psi}_g P_{Gg}^{\max} + c_{g0} - \mathbf{R}_g^{22}\right) \\ &- \sum_{g\in\mathcal{G}_i^{pw}} \sum_{t=1}^{r_g} \left(\zeta_{gt}\left(m_{gt}a_{gt} - b_{gt}\right)\right)\Bigg\} \\ &- \sum_{k\in\mathcal{L}} \left\{\left(S_k^{\max}\right)^2\left(\mathbf{H}_{k_l}^{11} + \mathbf{H}_{k_m}^{11}\right) + \mathbf{H}_{k_l}^{22} + \mathbf{H}_{k_m}^{22} + \mathbf{H}_{k_l}^{33} + \mathbf{H}_{k_m}^{33}\right\}\end{aligned} \tag{20}$$

The dual form of the semidefinite programing relaxation of the OPF problem is then written as

$$\max \quad \rho \tag{21a}$$

subject to

$$\mathbf{A} \succeq 0 \tag{21b}$$

$$\mathbf{H}_{k_l} \succeq 0, \quad \mathbf{H}_{k_m} \succeq 0 \qquad \forall k \in \mathcal{L} \tag{21c}$$

$$\mathbf{R}_g \succeq 0, \quad \mathbf{R}_g^{11} = 1 \qquad \forall g \in \mathcal{G}^q \tag{21d}$$

$$\sum_{t=1}^{r_g} \zeta_{gt} = 1 \qquad \forall g \in \mathcal{G}^{pw} \tag{21e}$$

$$\left\{\lambda_i = c_{g1} + 2\sqrt{c_{g2}}\mathbf{R}_g^{12} + \bar{\psi}_g - \underline{\psi}_g \quad \forall g \in \mathcal{G}_i^q\right\} \qquad \forall i \in \mathcal{N} \tag{21f}$$

$$\left\{\lambda_i = \sum_{t=1}^{r_g} \zeta_{gt} m_{gt} \qquad \forall g \in \mathcal{G}_i^{pw}\right\} \qquad \forall i \in \mathcal{N} \tag{21g}$$

$$\underline{\psi}_g \geq 0,\ \bar{\psi}_g \geq 0,\ \underline{\gamma}_i \geq 0,\ \bar{\gamma}_i \geq 0,\ \underline{\mu}_i \geq 0,\ \bar{\mu}_i \geq 0,\ \zeta_{gt} \geq 0 \tag{21h}$$

The semidefinite relaxation has zero duality gap and yields a physically meaningful solution if and only if the solution to (21) satisfies the rank condition $\dim\left(\mathrm{null}\left(\mathbf{A}\right)\right) \leq 2$.

C. Discussion

Several aspects of the semidefinite programming formulation deserve special attention. We focus on those aspects that differ from previous formulations (e.g., [5]) due to the

proposed formulation's allowing of multiple generators at the same bus and the possibility of parallel lines.

The semidefinite relaxation includes the possibility of multiple generators at the same bus. As shown in (21f) and (21g), all generators at the same bus i must have the same aggregate active power Lagrange multiplier λ_i. This is related to the principle of equal marginal costs in the economic dispatch problem [19]. Since generator reactive power injections do not appear in the cost function of (2), reactive power Lagrange multipliers are only needed for each generator bus rather than for each generator. This is seen in (17) and (18), which determine the allowed range of bus i reactive power injection.

The semidefinite relaxation also includes the possibility of parallel lines (i.e., multiple lines with the same terminal buses) and the ability to represent transformers with off-nominal voltage ratios and/or non-zero phase-shifts. Previous formulations limited line flows by constraining the total power flow between two buses, precluding the ability to separately limit line flows on parallel lines. This modeling flexibility comes at the price of additional complexity. Incorporating parallel lines removes the ability to form the line-flow matrices directly from the bus admittance matrix, instead requiring the more complicated expressions in (6)–(9). Incorporating off-nominal voltage ratios and non-zero phase shifts breaks the symmetry of the Π model such that different line-flow matrices are required for each line terminal (i.e., $\mathbf{Z}_{k_l}$ in (6) and $\bar{\mathbf{Z}}_{k_l}$ in (8) for active and reactive power flows measured from the sending terminal and $\mathbf{Z}_{k_m}$ in (7) and $\bar{\mathbf{Z}}_{k_m}$ in (9) for the receiving terminal).

For large system models, numerical difficulties in the semidefinite programming solver may prevent convergence to acceptable precision. We have found several pragmatic techniques that reduce numerical difficulties with large systems. First, ignore engineering limits that will clearly not be binding at the solution. Many power system data sets specify large values for limits that are intended to be unconstrained, particularly for reactive power generation and line-flow limits. We do not incorporate terms corresponding to very large limits. Similarly, some generators specified with quadratic cost functions actually have linear cost functions (i.e., $c_{g2} = 0$). The corresponding $\mathbf{R}_g$ matrix is eliminated. These techniques do not affect the optimality of the resulting solution.

Numerical difficulties often occur when the system model has very "tight" limits. For instance, the active power generation of a synchronous condenser is constrained to be zero. A second technique for reducing numerical difficulties is to use equality constraints rather than inequality constraints to model these limits. When the power output of a generator is constrained to a very small range, fix the generator at the midpoint of this range and directly add the associated generation cost to the objective function. The degree of suboptimality of the resulting solution can be estimated by multiplying the Lagrange multiplier corresponding to the equality constraint by the half of the difference between the maximum and minimum limits.

III. Advances in Matrix Completion Decompositions

In this section, we describe several advances in the decomposition techniques used to reduce the computational burden of semidefinite relaxations of large OPF problems. First, we review the maximal clique decomposition as introduced by Jabr [16]. Next, we present a matrix combination algorithm that significantly reduces the required computation time of this decomposition. We then discuss the rank condition properties of solutions to large system models. We next present a modification to Jabr's formulation of the maximal clique decomposition that extends this decomposition to general networks rather than only networks with admittance matrices that satisfy a definiteness requirement. Finally, we describe a technique for obtaining the optimal voltage profile from the decomposed matrices.

A. Overview of Jabr's Maximal Clique Decomposition

Jabr's formulation of the maximal clique decomposition uses a matrix completion theorem [13]. Several graph theoretic definitions aid understanding of this theorem. A "clique" is a subset of the graph vertices for which each vertex in the clique is connected to all other vertices in the clique. A "maximal clique" is a clique that is not a proper subset of another clique. A graph is "chordal" if each cycle of length four or more nodes has a chord, which is an edge connecting two nodes that are not adjacent in the cycle. The maximal cliques of a chordal graph can be determined in linear time [20]. See [16] and [21] for more details on these definitions.

The matrix completion theorem can now be stated. Let $\bar{\mathbf{A}}$ be a symmetric matrix with partial information (i.e., not all entries of $\bar{\mathbf{A}}$ have known values) with an associated undirected graph. Note that the graph of interest has the power system buses as vertices and the branch susceptances as edge weights. The matrix $\bar{\mathbf{A}}$ can be completed to a positive semidefinite matrix (i.e., the unknown entries of $\bar{\mathbf{A}}$ can be chosen such that $\bar{\mathbf{A}} \succeq 0$) if and only if the submatrices associated with each of the maximal cliques of the graph associated with $\bar{\mathbf{A}}$ are all positive semidefinite.

The matrix completion theorem allows replacing the single large $2n \times 2n$ positive semidefinite constraint (21b) by many smaller matrices that are each constrained to be positive semidefinite. This significantly reduces the problem size for large, sparse power networks.

There are two important aspects of this decomposition that are relevant to our advances. First, since the maximal cliques can have non-empty intersection (i.e., contain some of the same buses), different matrices may contain elements that refer to the same element in the $2n \times 2n$ matrix. Therefore, linking constraints are required to force equality between elements that are shared between maximal cliques. To specify these linking constraints, Jabr recommends forming a "clique tree": a maximum weight spanning tree of a graph with nodes corresponding to the maximal cliques and the edge weights between each node pair given by the number of shared buses in each clique pair. A maximal weight spanning tree of this graph, which can be calculated using Prim's algorithm [22],

is used to reduce the number of linking constraints: equality constraints are only enforced between the appropriate elements in maximal cliques that are adjacent in the maximal weight spanning tree.

Second, graphs corresponding to realistic power networks are not chordal. A chordal extension of the graph is thus required in order to use the matrix completion theorem. A chordal extension adds edges between non-physically connected nodes (i.e., edges in the chordal extension of the graph may exist between buses that are not connected by a line in the power system) to obtain a chordal graph. Jabr recommends obtaining a chordal extension using a Cholesky factorization of the absolute value of the imaginary part of the network's admittance matrix. To minimize the total number of edges, a Cholesky factorization with minimum fill-in is obtained from a minimum degree ordering of the row/column indices [23].

B. Matrix Combination Algorithm

We first describe a modification to the maximal clique decomposition that yields a significant computational speed improvement. This modification accounts for the trade-off between the size of maximal cliques and the number of linking constraints. Smaller maximal cliques generally reduce the total size of the positive semidefinite constrained matrices. The overlap between maximal cliques, as determined by the clique tree approach, establishes the number of linking constraints.

The maximal clique decomposition uses a Cholesky factorization with minimum fill-in to obtain small maximal cliques as a heuristic for minimizing the number of variables in the positive semidefinite matrix constraints. This approach does not account for the computational burden of the linking constraints. The optimization literature provides theoretical support for the concept of reducing computational burden by combining matrices (see [13] and Section 4 of [14]). Specifically, this literature discusses the potential trade off in solver time between the sizes of the semidefinite-constrained matrices and the number of linking constraints, which require solution of a system of linear equations in the semidefinite program algorithm. Combining matrices eliminates the need for linking constraints between the matrices at the computational price of a larger matrix. A heuristic for combining matrices to gain the benefits of small matrices while reducing linking constraints thus has the potential for computational speed improvements.

Many common semidefinite program solvers, such as SeDuMi [24] and SDPT3 [25], use primal–dual methods that solve both the primal and dual problems simultaneously. Moreover, since a primal constraint corresponds to a dual variable, the "size" of the semidefinite program can be estimated by adding the total number of scalar variables required to form the matrices with the number of linking constraints. This approximation for the "size" of a semidefinite program forms the basis of our matrix combination heuristic. In a greedy manner, we repeatedly combine the pair of matrices that most reduces the "size" of the semidefinite program as measured by the total number of variables plus the number of linking constraints.

We next detail our matrix combination heuristic. Let L be a parameter specifying the maximum number of matrices. Consider a semidefinite program formed from the chordal extension of a power system network, with maximal clique i containing d_i buses. Since the matrices corresponding to the maximal cliques are symmetric and contain both real and imaginary voltage components, matrix i (corresponding to maximal clique i) has $d_i\,(2d_i+1)$ variables. If maximal cliques i and k, adjacent in the clique tree, share s_{ik} buses, then $s_{ik}\,(2s_{ik}+1)$ linking constraints are required between the corresponding matrices. For each pair of adjacent maximal cliques in the clique tree, the change in the optimization problem "size" Δ_{ik} if the cliques i and k were combined is given by

$$\begin{aligned}\Delta_{ik} = {} & d_{ik}\,(2d_{ik}+1) - d_i\,(2d_i+1) \\ & - d_k\,(2d_k+1) - s_{ik}\,(2s_{ik}+1)\end{aligned} \tag{22}$$

where $d_{ik} = d_i + d_k - s_{ik}$ is the number of buses in the combined clique.

While the number of matrices is greater than L, combine a pair of adjacent maximal cliques with smallest Δ_{ik}. Then recalculate the value of Δ_{ik} for all maximal cliques adjacent to the newly combined clique. Repeat until the number of matrices is equal to L. Constrain the resulting set of matrices to be positive semidefinite in the OPF formulation.

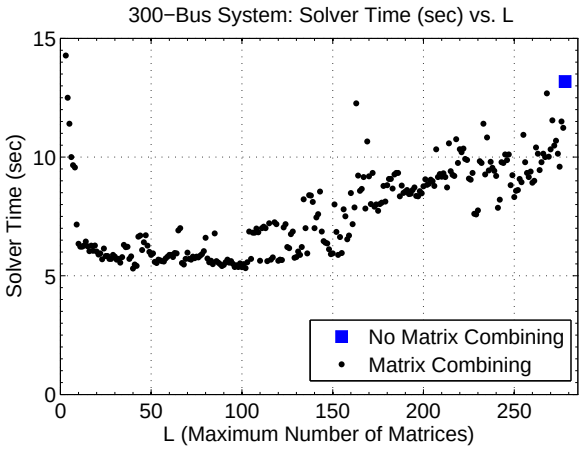
We test this heuristic using two large system models: the IEEE 300-bus system [26] and a 3012-bus model of the Polish system for evening peak demand in winter 2007-2008 [3]. These systems were chosen to examine how the heuristic scales with system size. Matrix decomposition techniques do not result in a notable speed improvement for small systems; no matrix decomposition significantly reduced the computational time for the IEEE 14, 30, and 57-bus systems [26].

The OPF formulation in (21) was implemented using YALMIP version 3 [27], SeDuMi version 1.3 [24], and MATLAB R2011a. One computer with an 64-bit Intel i7-2600 Quad Core CPU at 3.40 GHz with 16 GB of RAM was used to run the formulation. By using the matrix completion decompositions to exploit the inherent sparsity, these computational resources were sufficient for the 300 and 3012-bus system models. A tolerance of 1×10^{-9} for SeDuMi's `eps` was used in the calculation of the results.

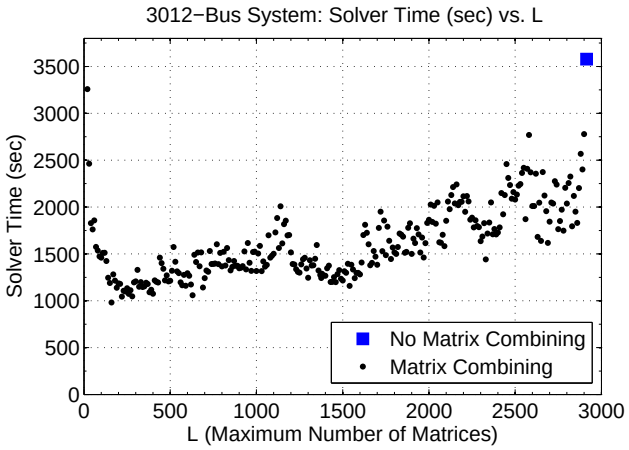
Figures 1a and 1b show variation in total solver time (i.e., the time used by the semidefinite programming solver SeDuMi), with the parameter L. These figures do not include the set up time required to initialize the formulation. However, the set up time is typically only 15% to 20% of the solver time.

The solver times for Jabr's formulation of the maximal clique decomposition are the rightmost points (no matrix combinations) in Figures 1a and 1b. As L decreases from the rightmost point, the solver times decrease by, at most, approximately a factor of 2.5 for the 300-bus system and a factor of 3.6 for the 3012-bus system as compared to the solver time without combining matrices.

The solution time graphs in Figures 1a and 1b appear to be "noisy," which we attribute to differences in the number of iterations needed to reach the specified tolerance. That is, solver times can vary among choices of L if it requires one



(a) Solver time vs. L for IEEE 300-Bus System. Note that the time with $L = 1$ of 69.5 seconds is not shown on the plot.



(b) Solver time vs. L for 3012-Bus System. Note that the time with $L = 2$ of approximately 1×10^5 seconds is not shown on the plot and the system could not be solved with $L = 1$.

Fig. 1. Solver Time vs. L

more solver iteration to reach the desired solution tolerance. Despite this noise, there is a clear trend. The plots show that reducing L results in significant improvements in solver time as compared to the case without matrix combining. However, as L continues to decrease, the speed improvements from removing linking constraints are overcome by the additional variables required for the larger matrices (in the extreme, returning to a single $2n \times 2n$ matrix for $L = 1$). Thus, the solver times exhibit a steep increase for small L. (Solution times for very small L are not shown on Figures 1a and 1b. For the 300-bus system, the full $2n \times 2n$ matrix constraint corresponding to $L = 1$ solved in 69.5 seconds. The 3012-bus system could not be solved with $L = 1$; the system required approximately 1.0×10^5 seconds for $L = 2$.)

Rather than combining matrices until below a specified parameter value, we also tried combining matrices until no pair of adjacent maximal cliques had a negative value of Δ_{ik} (i.e., stopping combining matrices once the heuristic indicated no further advantage to doing so). In our numerical experience, however, this approach typically did not identify a set of matrices that minimized the solver time. The number of matrices for which no remaining adjacent pairs of maximal cliques had negative Δ_{ik} was $L = 150$ for the 300-bus system and $L = 1376$ for the 3012-bus system. In both systems, faster solver times were obtained for smaller values of L. This reinforces the fact that our measure of semidefinite program size is a heuristic approximation of the computational burden.

Based on these empirical results, choosing L equal to 10% of the initial number of matrices appears to give near minimum solver times. (Expressing L as a percentage of the original number of matrices allows for easy comparison between systems.) A minimum computational time consistently appears at approximately this value of L for the available power system models; however, limited diversity of available large system models precludes more general comments on this value. We speculate that system models that are strongly interconnected (i.e., have a relatively low amount of sparsity) will inherently have large maximal cliques and thus not benefit from as many matrix combinations as compared to more sparsely connected system models. We therefore anticipate that a larger value of L would be appropriate for strongly interconnected models.

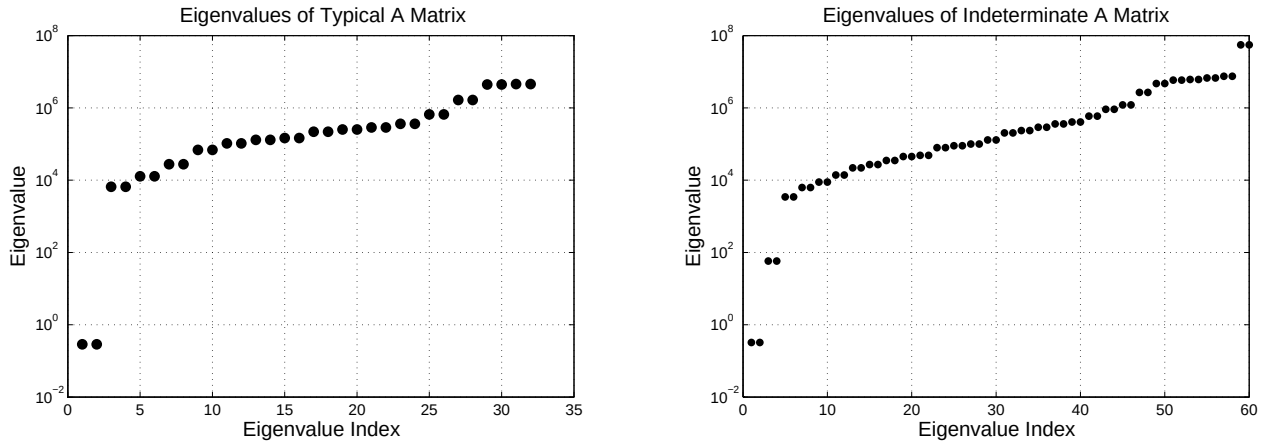
Note that solution times are not greatly affected by the addition of parallel lines or multiple generators at the same bus; models with these features have comparable computational burden relative to other models with the same total numbers of lines and generators.

Table I summarizes these results by providing the solver times for each system / decomposition pair along with a "speed up factor" (SUF) for the improvement of the matrix combination approach with $L = 10\%$ of the original number of matrices as compared to not combining matrices. Results using the full $2n \times 2n$ matrix for the 3012-bus system could not be computed. Note that solver times with $L = 10\%$ for other models of the Polish system that is represented in the 3012-bus system model are available in Table II. Also note that the proposed heuristic yields improvements for the intermediate-sized 118-bus system.

System	$2n \times 2n$	No Combining	Combining ($L = 10\%$)	SUF
IEEE 118-bus	6.63	4.84	2.06	2.349
IEEE 300-bus	69.45	13.18	5.71	2.309
Polish 3012-bus	–	3578.5	1197.4	2.989

TABLE I
SOLVER TIMES (SEC) FOR VARIOUS ALGORITHMS

We attempted several alternatives to the proposed heuristic: a variant of the proposed algorithm that, at each step, randomly (weighted by Δ_{ik}) selects a pair of maximal cliques to combine; the heuristic proposed in [14]; and a "top-down" approach that groups maximal cliques using a normalized cut algorithm on the clique tree. As compared to the proposed heuristic, these alternatives sometimes had comparable, but not faster, solution times.



(a) Eigenvalues for a Typical $\mathbf{A}$ Matrix

(b) Eigenvalues for an Indeterminate $\mathbf{A}$ Matrix

Fig. 2. Eigenvalues for Selected $\mathbf{A}$ Matrices of the Solution to the 3012-Bus System

C. Analysis of Duality Gap Properties of Solution to OPF Problems for Large System Models

While solutions to many OPF problems satisfy the rank condition and thus have zero duality gaps [5], it is known that some small system models yield solutions with non-zero duality gaps [6], [7]. Until the recent exploitation of power system sparsity, computational challenges have precluded investigation of the rank properties of the semidefinite relaxation for large system models. Harnessing the computational methods described in this paper, we conduct further investigation of rank condition satisfaction for large system models. Note that, as in the previous section, the results in this section are calculated with a minimum line resistance of 1×10^{-4} per unit in accordance with [5] and with SeDuMi's tolerance parameter `eps` set to 1×10^{-9}.

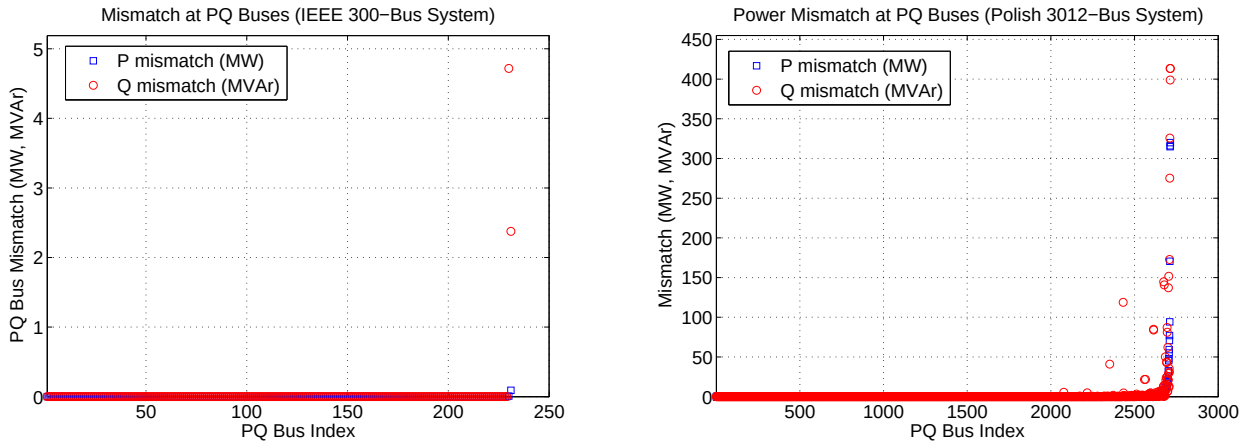
When using a matrix completion decomposition, solutions to the dual formulation of the semidefinite relaxation consist of a set of $\mathbf{A}$ matrices. For a solution that satisfies the rank condition, the nullspaces of all $\mathbf{A}$ matrices have dimension less than or equal to two. However, for numerical reasons, solvers do not yield a "hard zero" value for the eigenvalues corresponding to the nullspaces of these matrices. Therefore, it can be difficult to determine when an $\mathbf{A}$ matrix has nullspace with dimension two. For illustration of this challenge, Figure 2 shows the eigenvalues, sorted in order of ascending magnitude, for selected $\mathbf{A}$ matrices from the $L = 10\%$ decomposition of the Polish 3012-bus system model. With two smallest eigenvalues that are four orders of magnitude below the next smallest eigenvalues, Figure 2a shows a typical matrix that has nullspace with dimension two. Conversely, the smallest eigenvalues in Figure 2b are only two orders of magnitude below the next smallest eigenvalues; the nullspace dimension for this matrix is more difficult to determine. Characterizing the overall satisfaction of the rank condition for the Polish 3012-bus system is correspondingly difficult.

To evaluate the satisfaction of the rank condition for methods that exploit power system sparsity by using matrix completion decompositions, we propose the following metric to measure closeness to a nullspace with dimension two. The metric is based on the ratio between the third and second smallest magnitude eigenvalues. The minimum such ratio among all the $\mathbf{A}$ matrices is termed the "minimum eigenvalue ratio." If the solution did yield "hard zeros" for zero eigenvalues, the second smallest eigenvalue would be zero and the third smallest eigenvalue would be non-zero, resulting in a minimum eigenvalue ratio of infinity. In practice, numerical issues result in minimum eigenvalue ratios that are large (typical values are greater than 1×10^7 for small systems that are known to satisfy the rank condition). Further, if the solution does not satisfy the rank condition, both the second and third smallest eigenvalues will have similar magnitudes near zero, therefore yielding a small value for the minimum eigenvalue ratio. Thus, a large value for the minimum eigenvalue ratio indicates a solution with zero duality gap while a small value indicates a non-zero duality gap solution. Note that more complex metrics than the proposed minimum eigenvalue ratio are possible; the proposed metric is intended to be a simple but meaningful measure.

Table II shows the minimum eigenvalue ratios for several test systems. The solution times with $L = 10\%$ are also given. The systems with more than 300 buses are representations of the Polish grid with various levels of modeling detail and different loading scenarios (winter peak (wp), winter off peak

System Model	Min Eigenvalue Ratio	Max Mismatch	Solver Time ($L = 10\%$)
IEEE 118-bus	2.86×10^9	3.9×10^{-5} MVAr	2.1 sec
IEEE 300-bus	2.25×10^2	4.7×10^0 MVAr	5.7 sec
2383-bus (wp)	7.90×10^2	2.9×10^2 MVAr	730 sec
2736-bus (sp)	3.07×10^4	2.7×10^{-2} MVAr	622 sec
2737-bus (sop)	4.11×10^4	3.7×10^{-1} MVAr	607 sec
2746-bus (wp)	8.65×10^4	5.5×10^{-2} MW	752 sec
2746-bus (wop)	1.95×10^4	1.4×10^{-1} MW	738 sec
3012-bus (wp)	1.72×10^2	4.1×10^2 MVAr	1197 sec
3120-bus (sp)	5.84×10^2	4.6×10^1 MVAr	1619 sec
3375-bus (wp)	1.64×10^2	5.2×10^2 MVAr	1457 sec

TABLE II
MEASURES OF RANK CONDITION SATISFACTION AND SOLVER TIMES FOR VARIOUS SYSTEM MODELS



(a) Power Mismatch for IEEE 300-Bus System

(b) Power Mismatch for Polish 3012-Bus System

Fig. 3. Active and Reactive Power Mismatch at PQ Buses

(wop), summer peak (sp), and summer off peak (sop)). These results indicate that, according to the proposed metric, the large system models do not satisfy the rank condition as well as many smaller system models, which generally have minimum eigenvalue ratios greater than 1×10^7 when they satisfy the rank condition. Since, other than the IEEE 118 and 300-bus systems, the large system models all represent the same Polish system, the lack of more diverse system models limits the ability to make more general statements concerning satisfaction of the rank condition for large system models.

An alternative test for satisfaction of the rank condition is based on the mismatch between the calculated and specified active and reactive power injections at PQ buses. To recover a candidate voltage profile, we form the closest rank one matrix to the solution's $\mathbf{W}$ matrix using the eigenvector associated with the largest eigenvalue of $\mathbf{W}$. If the solution has zero duality gap, the matrix $\mathbf{W}$ is rank one and the resulting voltage profile will satisfy the power injection equality constraints at the PQ buses. Conversely, the closest rank one matrix to a solution with non-zero duality gap will typically not yield a voltage profile that satisfies the power injection equality constraints at PQ buses. Thus, the mismatch between the calculated and specified power injections at PQ buses provides an alternative measure for satisfaction of the rank condition.

Figures 3a and 3b show the mismatch between the specified and calculated active and reactive power injections at PQ buses for the 300-bus and 3012-bus systems, respectively, sorted in order of increasing active power mismatch. The voltage profile yields small mismatches for the majority of buses, but a few buses display large mismatches in both active and reactive power. The large power mismatches indicate solutions having non-zero duality gap. With small mismatch at the majority of PQ buses, such solutions with non-zero duality gap may provide good starting points for a local search algorithm. Table II shows the maximum mismatch, considering both active and reactive powers, for a variety of test systems. Solutions to several of these system models have relatively large power mismatches; for instance, mismatches for all test systems in Table II except for the 118, 2736, and 2746 (wp) bus systems are greater than the default Newton solution tolerance of 0.1 MW/MVAr used by the power flow solution package PSS/E [28]. Large power mismatches indicate that the corresponding solutions do not satisfy the rank condition. Note the correlation between the minimum eigenvalue ratio and the maximum power mismatch, which supports the validity of these measures of rank condition satisfaction.

D. Extending Jabr's Formulation of the Maximal Clique Decomposition to All Systems

The first step in Jabr's formulation creates a chordal extension of the network using a Cholesky factorization of the absolute value of the imaginary part of the bus admittance matrix (i.e., $\mathrm{chol}\left(\left|\mathrm{Im}\left(\mathbf{Y}\right)\right|\right)$). Only positive definite matrices have Cholesky factorizations. Since not all power system networks have admittance matrices that satisfy $\left|\mathrm{Im}\left(\mathbf{Y}\right)\right| \succ 0$ (e.g., networks with sufficiently large shunt capacitances), Jabr's formulation cannot be universally applied to such networks.

Jabr's formulation only uses the *sparsity pattern* (i.e., location of the non-zero elements) of the Cholesky factorization. Thus, an alternative, positive definite matrix whose Cholesky factorization exhibits the same sparsity pattern would extend Jabr's formulation to general power systems. We next present such an alternative matrix.

Let $\mathbf{D}$ represent the incidence matrix associated with the network (i.e., each row of $\mathbf{D}$ corresponds to a line and has two non-zero elements: $+1$ in the column corresponding to the line's "from" bus and -1 in the column corresponding to the line's "to" bus). The matrix $\mathbf{E}$ in (23) has a Cholesky factorization with the same sparsity pattern as $\mathrm{chol}\left(\left|\mathrm{Im}\left(\mathbf{Y}\right)\right|\right)$.

$$\mathbf{E} = \mathbf{D}^T\mathbf{D} + \mathbf{I}_{n\times n} \tag{23}$$

where $\mathbf{I}_{n\times n}$ is the $n \times n$ identity matrix.

Since $\mathbf{D}^T\mathbf{D}$ has a Laplacian structure, it is positive semidefinite. Adding an identity matrix increases all eigenvalues by one, and thus $\mathbf{E}$ is positive definite. Note that the common modification for making a Laplacian matrix positive definite via adding the matrix $\mathbb{1}\cdot\mathbb{1}^T$, where $\mathbb{1}$ is the vector of all ones,

is not appropriate due to the fact that this modification makes the Cholesky factorization of $\mathbf{E}$ dense.

The bus admittance matrix $\mathbf{Y}$ has generalized Laplacian structure, with weightings from the line admittances, plus diagonal terms corresponding to shunt admittances. The $\mathbf{E}$ matrix's similar construction implies that its Cholesky factorization has the same sparsity pattern as the Cholesky factorization of $|\mathrm{Im}\,(\mathbf{Y})|$. Using the Cholesky factorization of $\mathbf{E}$ therefore extends Jabr's method to general power networks.

E. Obtaining the Optimal Voltage Profile

The solution to a decomposed problem is a set of positive semidefinite matrices. If all the matrices have nullspaces with appropriate dimension, the optimal voltage profile can be recovered [5], [16]. (For formulations that separate real and imaginary voltage components, like (21), the nullspace of all matrices must have dimension less than or equal to two.) However, existing literature does not give a detailed method for recovering the optimal voltage profile. We next describe a technique for obtaining the optimal voltage profile.

An overview of this technique follows. First obtain vectors in the nullspaces (hereafter referred to as nullvectors) of each positive semidefinite constrained matrix. Note that calculation of these nullvectors can be carried out in parallel since the nullspace computation for each matrix can be performed independently. These nullvectors, when rearranged such that they correspond to complex "phasor" voltages, can each be multiplied by a different complex scalar and remain in their respective nullspaces. Since a bus can be in multiple maximal cliques, elements in different vectors may correspond to the same bus voltage phasor. The complex scalars are chosen such that elements of different vectors that correspond to the same bus voltage are equal. A centrally computed linear nullspace calculation of a specified matrix gives an appropriate choice of the scalar values. This allows for specification of a vector that is a real scalar multiple of the optimal voltage profile. Using a single binding constraint, the resulting vector is scaled to obtain the optimal voltage profile.

We next present the details of this technique. Consider an optimal solution to (21) consisting of d positive semidefinite matrices $\bar{\mathbf{A}}_i$ with $\dim\left(\mathrm{null}\left(\bar{\mathbf{A}}_i\right)\right) \leq 2,\ \forall i \in \{1,\ldots,d\}$. Let $u^{(i)}$ be a nullvector of $\bar{\mathbf{A}}_i$. Let r_i be the number of buses in the maximal clique corresponding to matrix i. Convert each vector $u^{(i)}$ to complex "phasor" form: $\underline{u}^{(i)} = u^{(i)}_{1:r_i} + j u^{(i)}_{r_i+1:2r_i}$, where subscript $1:r_i$ indicates the first through r_i^{th} elements of the corresponding vector.

Vectors $\underline{u}^{(i)}$ remain in their corresponding nullspace after multiplication by complex scalars α_i. This property is used to enforce consistency between elements of different vectors that correspond to the same bus voltage phasor. Obtaining the optimal voltage profile requires determining values of α_i that create agreement between all elements representing the same voltage from the nullvectors of different matrices. This can be visualized by forming a table with rows corresponding to bus indices and columns corresponding to maximal clique indices. If maximal clique j contains bus i, the (i,j) entry of the table is α_j multiplied by the element of $\underline{u}^{(j)}$ corresponding to bus i. If maximal clique j does not contain bus i, the (i,j) entry of the table is empty.

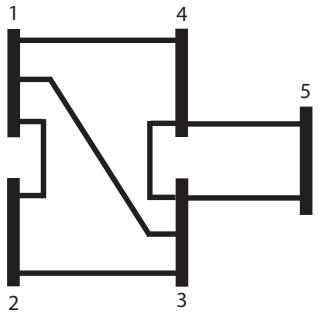


Fig. 4. Illustrative Voltage Profile Recovery Example Network

Bus \ Clique Index	1	2	3
1	$\alpha_1 \underline{u}_1^{(1)}$	$\alpha_2 \underline{u}_1^{(2)}$	
2	$\alpha_1 \underline{u}_2^{(1)}$		
3	$\alpha_1 \underline{u}_3^{(1)}$	$\alpha_2 \underline{u}_2^{(2)}$	$\alpha_3 \underline{u}_1^{(3)}$
4		$\alpha_2 \underline{u}_3^{(2)}$	$\alpha_3 \underline{u}_2^{(3)}$
5			$\alpha_5 \underline{u}_3^{(3)}$

TABLE III
ILLUSTRATIVE VOLTAGE PROFILE RECOVERY EXAMPLE TABLE

Since each row of the table represents a voltage phasor at the corresponding bus, values of $\alpha_i\ \forall i = 1,\ldots,d$ are chosen such that all entries in each row are equal. Appropriate values of α_i are obtained using a nullvector of an appropriately specified matrix. Specifically, use the following procedure to create a matrix $\mathbf{C}$ with d columns that enforces equality of all entries of each row of the table. For each row i of the table, find the first non-empty entry and store the corresponding column index j. (All rows of the table will have at least one non-empty entry because each bus is contained in at least one maximal clique.) While there exists a non-empty entry in row i with column index greater than j (let the non-empty entry exist in column k), add a row to the matrix $\mathbf{C}$ that enforces equality of the (i,j) and (i,k) entries. Set $j = k$ and repeat until no other non-empty entries exist in row i with column indices greater than j. Then proceed to row $i+1$.

Consider the illustrative example system network in Figure 4 and corresponding Table III. This system has three maximal cliques composed of buses $\{1, 2, 3\}$, $\{1, 3, 4\}$, and $\{3, 4, 5\}$. The corresponding equation for the example is

$$\mathbf{C}\alpha = \begin{bmatrix} \underline{u}_1^{(1)} & -\underline{u}_1^{(2)} & 0 \\ \underline{u}_3^{(1)} & -\underline{u}_2^{(2)} & 0 \\ 0 & \underline{u}_2^{(2)} & -\underline{u}_1^{(3)} \\ 0 & \underline{u}_3^{(2)} & -\underline{u}_2^{(3)} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (24)$$

The nullspace calculation has a non-trivial solution if all $\bar{\mathbf{A}}_i$ matrices of the solution have nullspaces with dimension less than or equal to two. (For a solution to the semidefinite relaxation where some of the $\bar{\mathbf{A}}$ matrices have nullspace dimension greater than two, the nullspace calculation may only

have the trivial solution $\alpha = 0$, indicating that a consistent voltage profile cannot be obtained.) A nullvector α yields a table where all entries of each row have the same value. Create a vector η of length n where η_i is equal to the value of an entry in the i^{th} row of the table. The vector η is a scalar multiple of the optimal voltage vector.

Since α has one degree of freedom in its length, the optimal voltage profile is a real scalar multiple χ of η. To determine the value of χ, one additional piece of information is required from a binding constraint. Reference [5] suggests the use of a binding voltage magnitude constraint. However, not all solutions have a binding voltage magnitude constraint (e.g., the three-bus system in [6]). Optimal solutions to OPF problems have at least one binding constraint, but not necessarily a binding voltage magnitude constraint.

A binding constraint is identified by a non-zero value of the corresponding Lagrange multiplier. Consider a solution with a binding voltage magnitude constraint. Let $\bar{V}_k$ be the value of a binding voltage magnitude constraint at bus k. The value of χ is chosen using this voltage magnitude:

$$\chi = \frac{\bar{V}_k}{|\eta_k|} \tag{25}$$

For solutions without a binding voltage magnitude constraint, use an alternative binding constraint to determine χ.

The optimal voltage profile is then constructed by scaling η by χ and rotating the resulting vector to obtain zero reference angle.

$$V^{opt} = \chi \eta e^{-j\theta_{ref}} \tag{26}$$

where θ_{ref} is the angle of the element of η corresponding to the reference bus.

IV. Conclusion and Future Work

This paper has addressed two categories of practical issues associated with implementing a large-scale optimal power flow solver based on semidefinite programming: modeling issues associated with general power systems and computational issues. Specific modeling issues addressed include multiple generators at the same bus and limiting flows on parallel lines. Both quadratic and convex piecewise-linear generator cost functions are considered.

The paper provides three computational advances for exploiting power system sparsity using matrix completion decompositions. First, a proposed matrix combination algorithm considers the impact of "linking constraints" between elements in certain decomposed matrices that refer to the same element in the original $2n \times 2n$ matrix. Since combining matrices eliminates linking constraints, matrix combination can reduce computation time. Calculations using test systems show the efficacy of the matrix combination approach: the IEEE 300-bus system shows a factor of approximately 2.3 decrease in solver time and a 3012-bus model of the Polish system shows a factor of 3.0 decrease in solver time compared to not combining matrices. The rank characteristics of solutions to OPF problems for large system models are also examined.

Next, Jabr's formulation of the maximal clique decomposition [16] was extended to general power system networks. This formulation uses a Cholesky factorization of the absolute value of the imaginary part of the bus admittance matrix. Since a Cholesky factorization requires a positive definite matrix, this approach cannot be used for some networks (e.g., networks with large shunt capacitive compensation). Jabr's formulation only uses the sparsity pattern of the Cholesky factorization. We propose an alternative positive definite matrix with the same sparsity pattern to extend Jabr's formulation to general power system networks.

A final computational advance is a method for constructing the optimal voltage profile from a solution consisting of decomposed matrices. Although existing literature discusses the use of matrix decompositions [10], [16], [17], it does not give a detailed method for obtaining the optimal voltage profile.

Future work includes investigation of alternative load models. Currently, the formulation includes the capability for constant power and constant impedance load models. Another common load model is constant current, which is not trivially incorporated into the semidefinite programming formulation. Investigation of whether a constant current model can be included in a semidefinite programming-based OPF solver is thus valuable.

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References

[1] B. Lesieutre and I. Hiskens, "Convexity of the Set of Feasible Injections and Revenue Adequacy in FTR Markets," *IEEE Transactions on Power Systems*, vol. 20, no. 4, pp. 1790 – 1798, November 2005.

[2] Z. Qiu, G. Deconinck, and R. Belmans, "A Literature Survey of Optimal Power Flow Problems in the Electricity Market Context," in *Power Systems Conference and Exposition, 2009. PSCE '09. IEEE/PES*, March 2009, pp. 1–6.

[3] R. Zimmerman, C. Murillo-Sánchez, and R. Thomas, "MATPOWER: Steady-State Operations, Planning, and Analysis Tools for Power Systems Research and Education," *IEEE Transactions on Power Systems*, no. 99, pp. 1–8, 2011.

[4] X. Bai, H. Wei, K. Fujisawa, and Y. Wang, "Semidefinite Programming for Optimal Power Flow Problems," *International Journal of Electrical Power & Energy Systems*, vol. 30, no. 6-7, pp. 383–392, 2008.

[5] J. Lavaei and S. Low, "Zero Duality Gap in Optimal Power Flow Problem," *IEEE Transactions on Power Systems*, vol. 27, no. 1, pp. 92–107, Feb. 2012.

[6] B. C. Lesieutre, D. K. Molzahn, A. R. Borden, and C. L. DeMarco, "Examining the Limits of the Application of Semidefinite Programming to Power Flow Problems," in *49th Annual Allerton Conference on Communication, Control, and Computing, 2011*, Sept. 28-30 2011.

[7] W. A. Bukhsh, A. Grothey, K. I. McKinnon, and P. A. Trodden, "Local Solutions of Optimal Power Flow," University of Edinburgh School of Mathematics, Tech. Rep. ERGO 11-017, 2011, [Online]. Available: http://www.maths.ed.ac.uk/ERGO/pubs/ERGO-11-017.html.

[8] B. Zhang and D. Tse, "Geometry of Feasible Injection Region of Power Networks," in *49th Annual Allerton Conference on Communication, Control, and Computing, 2011*, Sept. 28-30 2011.

[9] S. Bose, D. Gayme, S. Low, and K. Chandy, "Optimal Power Flow Over Tree Networks," in *49th Annual Allerton Conference on Communication, Control, and Computing, 2011*, Sept. 28-30 2011.

[10] S. Sojoudi and J. Lavaei, "Physics of Power Networks Makes Hard Optimization Problems Easy To Solve," in *2012 IEEE Power & Energy Society General Meeting*, July 22-27 2012.

[11] J. Lavaei, D. Tse, and B. Zhang, "Geometry of Power Flows in Tree Networks," in *2012 IEEE Power & Energy Society General Meeting*, July 22-27 2012.

[12] D. K. Molzahn, B. C. Lesieutre, and C. L. DeMarco, "A Sufficient Condition for Power Flow Insolvability with Applications to Voltage Stability Margins," To appear in *IEEE Transactions on Power Systems*.

[13] M. Fukuda, M. Kojima, K. Murota, K. Nakata *et al.*, "Exploiting Sparsity in Semidefinite Programming via Matrix Completion I: General Framework," *SIAM Journal on Optimization*, vol. 11, no. 3, pp. 647–674, 2001.

[14] K. Nakata, K. Fujisawa, M. Fukuda, M. Kojima, and K. Murota, "Exploiting Sparsity in Semidefinite Programming via Matrix Completion II: Implementation and Numerical Results," *Mathematical Programming*, vol. 95, no. 2, pp. 303–327, 2003.

[15] S. Kim, M. Kojima, M. Mevissen, and M. Yamashita, "Exploiting Sparsity in Linear and Nonlinear Matrix Inequalities via Positive Semidefinite Matrix Completion," *Mathematical Programming*, vol. 129, no. 1, pp. 33–68, 2011.

[16] R. A. Jabr, "Exploiting Sparsity in SDP Relaxations of the OPF Problem," *IEEE Transactions on Power Systems*, vol. PP, no. 99, p. 1, 2011.

[17] X. Bai and H. Wei, "A Semidefinite Programming Method with Graph Partitioning Technique for Optimal Power Flow Problems," *International Journal of Electrical Power & Energy Systems*, 2011.

[18] A. Lam, B. Zhang, and D. Tse, "Distributed algorithms for optimal power flow problem," *arXiv preprint arXiv:1109.5229*, 2011.

[19] J. Glover, M. Sarma, and T. Overbye, *Power System Analysis and Design*. Thompson Learning, 2008.

[20] R. Tarjan and M. Yannakakis, "Simple Linear-Time Algorithms to Test Chordality of Graphs, Test Acyclicity of Hypergraphs, and Selectively Reduce Acyclic Hypergraphs," *SIAM Journal on Computing*, vol. 13, p. 566, 1984.

[21] G. Valiente, *Algorithms on Trees and Graphs*. Springer Verlag, 2002.

[22] J. Gross and J. Yellen, *Graph Theory and its Applications*. CRC press, 2006.

[23] P. Amestoy, T. Davis, and I. Duff, "Algorithm 837: AMD, An Approximate Minimum Degree Ordering Algorithm," *ACM Transactions on Mathematical Software (TOMS)*, vol. 30, no. 3, pp. 381–388, 2004.

[24] J. Sturm, "Using SeDuMi 1.02, A MATLAB Toolbox for Optimization Over Symmetric Cones," *Optimization Methods and Software*, vol. 11, no. 1, pp. 625–653, 1999.

[25] R. Tütüncü, K. Toh, and M. Todd, "Solving Semidefinite-Quadratic-Linear Programs using SDPT3," *Mathematical Programming*, vol. 95, no. 2, pp. 189–217, 2003.

[26] Power Systems Test Case Archive. University of Washington Department of Electrical Engineering. [Online]. Available: http://www.ee.washington.edu/research/pstca/

[27] J. Lofberg, "YALMIP: A Toolbox for Modeling and Optimization in MATLAB," in *IEEE International Symposium on Computer Aided Control Systems Design, 2004*. IEEE, 2004, pp. 284–289.

[28] Siemens PTI, "OPF Manual," *Power System Simulation for Engineering (PSS/E)*, vol. 31.0, December 2007.

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