AC Power Flows and their Derivatives using Complex Matrix Notation and Cartesian Coordinate Voltages

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MATPOWER *Technical Note 4*

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Contents

1 Notation

n_b, n_g, n_l, n_r	number of buses, generators, branches, reference buses, respectively
u_i, w_i	real and imaginary parts of bus voltage at bus i
$\|v_i\|, \theta_i$	bus voltage magnitude and angle at bus i
v_i	complex bus voltage at bus i, that is $\|v_i\|e^{j\theta_i}$ or $u_i + jw_i$
U, W	$n_b \times 1$ vectors of real and imaginary parts of bus voltage
$\mathcal{V}, \Theta$	$n_b \times 1$ vectors of bus voltage magnitudes and angles
V	$n_b \times 1$ vector of complex bus voltages v_i, $U + jW$
I^{bus}	$n_b \times 1$ vector of complex bus current injections
I^f, I^t	$n_l \times 1$ vectors of complex branch current injections, *from* and *to* ends
S^{bus}	$n_b \times 1$ vector of complex bus power injections
S^f, S^t	$n_l \times 1$ vectors of complex branch power flows, *from* and *to* ends
S_g	$n_g \times 1$ vector of generator complex power injections
P, Q	real and reactive power flows/injections, $S = P + jQ$
M, N	real and imaginary parts of current flows/injections, $I = M + jN$
Y_{bus}	$n_b \times n_b$ system bus admittance matrix
Y_f, Y_t	$n_l \times n_b$ system branch admittance matrices, *from* and *to* ends
C_g	$n_b \times n_g$ generator connection matrix $(i,j)^{th}$ element is 1 if generator j is located at bus i, 0 otherwise
C_f, C_t	$n_l \times n_b$ branch connection matrices, *from* and *to* ends, $(i,j)^{th}$ element is 1 if *from* end, or *to* end, respectively, of branch i is connected to bus j, 0 otherwise
C_{ref}	$n_r \times n_b$ reference bus indicator matrix $(i,j)^{th}$ element is 1 if bus j is i^{th} reference bus, 0 otherwise
$[A]$	diagonal matrix with vector A on the diagonal
A^T	(non-conjugate) transpose of matrix A
A^*	complex conjugate of A
A^b	matrix exponent for matrix A, or element-wise exponent for vector A
$\mathbf{1}_n, [\mathbf{1}_n]$	$n \times 1$ vector of all ones, $n \times n$ identity matrix
$\mathbf{0}$	appropriately-sized vector or matrix of all zeros

2 Introduction

This document is a companion to MATPOWER *Technical Note 2* [1] and MATPOWER *Technical Note 3* [2]. The purpose of these documents is to show how the AC power balance and flow equations used in power flow and optimal power flow computations can be expressed in terms of complex matrices, and how their first and second derivatives can be computed efficiently using complex sparse matrix manipulations. The relevant code in MATPOWER [3,4] is based on the formulas found in these three notes.

MATPOWER *Technical Note 2* presents a standard formulation based on complex power flows and nodal power balances using a polar representation of bus voltages, MATPOWER *Technical Note 3* adds the formulas needed for nodal current balances, and this note presents versions of both based on a cartesian coordinate representation of bus voltages.

We will be looking at complex functions of the real valued vector

$$X = \begin{bmatrix} U \\ W \\ P_g \\ Q_g \end{bmatrix}. \tag{1}$$

For a complex scalar function $f\colon \mathbb{R}^n \to \mathbb{C}$ of a real vector $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\mathsf{T}$, we use the following notation for the first derivatives (transpose of the gradient)

$$f_X = \frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}. \tag{2}$$

The matrix of second partial derivatives, the Hessian of f, is

$$f_{XX} = \frac{\partial^2 f}{\partial X^2} = \frac{\partial}{\partial X}\left(\frac{\partial f}{\partial X}\right)^\mathsf{T} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}. \tag{3}$$

For a complex vector function $F\colon \mathbb{R}^n \to \mathbb{C}^m$ of a vector X, where

$$F(X) = \begin{bmatrix} f_1(X) & f_2(X) & \cdots & f_m(X) \end{bmatrix}^\mathsf{T}, \tag{4}$$

the first derivatives form the Jacobian matrix, where row i is the transpose of the gradient of f_i.

$$F_X = \frac{\partial F}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{5}$$

In these derivations, the full 3-dimensional set of second partial derivatives of F will not be computed. Instead a matrix of partial derivatives will be formed by computing the Jacobian of the vector function obtained by multiplying the transpose of the Jacobian of F by a constant vector λ, using the following notation.

$$F_{XX}(\alpha) = \left(\frac{\partial}{\partial X} \left({F_X}^\mathsf{T} \lambda \right) \right) \bigg|_{\lambda=\alpha} \tag{6}$$

Just to clarify the notation, if Y and Z are subvectors of X, then

$$F_{YZ}(\alpha) = \left(\frac{\partial}{\partial Z} \left({F_Y}^\mathsf{T} \lambda \right) \right) \bigg|_{\lambda=\alpha}. \tag{7}$$

One common operation encountered in these derivations is the element-wise multiplication of a vector A by a vector B to form a new vector C of the same dimension, which can be expressed in either of the following forms

$$C = [A]\, B = [B]\, A \tag{8}$$

It is useful to note that the derivative of such a vector can be calculated by the chain rule as

$$C_X = \frac{\partial C}{\partial X} = [A] \frac{\partial B}{\partial X} + [B] \frac{\partial A}{\partial X} = [A]\, B_X + [B]\, A_X \tag{9}$$

3 Voltages

3.1 Bus Voltages

V is the $n_b \times 1$ vector of complex bus voltages. The element for bus i is $v_i = u_i + j w_i$. U and W are the vectors of real and imaginary parts of the bus voltages. Consider also the vector of inverses of bus voltages $\frac{1}{v_i}$, denoted by Λ. Note that

$$\frac{1}{v_i} = \frac{1}{u_i + j w_i} = \frac{u_i - j w_i}{u_i^2 + w_i^2} = \frac{{v_i}^*}{|v_i|^2} \tag{10}$$

$$\Lambda = V^{-1} = [\mathcal{V}]^{-2}\, V^* \tag{11}$$

$$\Theta = \tan^{-1}\left([U]^{-1}\, W \right) \tag{12}$$

$$\mathcal{V} = \left(U^2 + W^2 \right)^{\frac{1}{2}} \tag{13}$$

3.1.1 First Derivatives

$$V_U = \frac{\partial V}{\partial U} = [\mathbf{1}_{n_b}] \tag{14}$$

$$V_W = \frac{\partial V}{\partial W} = j\,[\mathbf{1}_{n_b}] \tag{15}$$

$$\Lambda_U = \frac{\partial \Lambda}{\partial U} = -\,[\Lambda]^2 \tag{16}$$

$$\Lambda_W = \frac{\partial \Lambda}{\partial W} = -j\,[\Lambda]^2 \tag{17}$$

The following could also be useful for implementing certain constraints on voltage magnitude or angles. For the derivations, see the scalar versions found in Appendix A.

$$\Theta_U = \frac{\partial \Theta}{\partial U} = -\,[\mathcal{V}]^{-2}\,[W] \tag{18}$$

$$\Theta_W = \frac{\partial \Theta}{\partial W} = [\mathcal{V}]^{-2}\,[U] \tag{19}$$

$$\mathcal{V}_U = \frac{\partial \mathcal{V}}{\partial U} = [\mathcal{V}]^{-1}\,[U] \tag{20}$$

$$\mathcal{V}_W = \frac{\partial \mathcal{V}}{\partial W} = [\mathcal{V}]^{-1}\,[W] \tag{21}$$

3.1.2 Second Derivatives

For the derivations, see the scalar versions found in Appendix A.

$$\Theta_{UU}(\lambda) = \frac{\partial}{\partial U}\left({\Theta_U}^{\mathsf{T}}\lambda\right) \tag{22}$$

$$= 2\,[\lambda]\,[\mathcal{V}]^{-4}\,[U]\,[W] \tag{23}$$

$$\Theta_{UW}(\lambda) = \frac{\partial}{\partial W}\left({\Theta_U}^{\mathsf{T}}\lambda\right) \tag{24}$$

$$= [\lambda] [\mathcal{V}]^{-4} \left([W]^2 - [U]^2\right) \tag{25}$$

$$\Theta_{WU}(\lambda) = \frac{\partial}{\partial U} \left(\Theta_W{}^{\mathsf{T}} \lambda\right) \tag{26}$$
$$= [\lambda] [\mathcal{V}]^{-4} \left([W]^2 - [U]^2\right) \tag{27}$$

$$\Theta_{WW}(\lambda) = \frac{\partial}{\partial W} \left(\Theta_W{}^{\mathsf{T}} \lambda\right) \tag{28}$$
$$= -2 [\lambda] [\mathcal{V}]^{-4} [U] [W] \tag{29}$$

$$\mathcal{V}_{UU}(\lambda) = \frac{\partial}{\partial U} \left(\mathcal{V}_U{}^{\mathsf{T}} \lambda\right) \tag{30}$$
$$= [\lambda] [\mathcal{V}]^{-3} [W]^2 \tag{31}$$

$$\mathcal{V}_{UW}(\lambda) = \frac{\partial}{\partial W} \left(\mathcal{V}_U{}^{\mathsf{T}} \lambda\right) \tag{32}$$
$$= - [\lambda] [\mathcal{V}]^{-3} [U] [W] \tag{33}$$

$$\mathcal{V}_{WU}(\lambda) = \frac{\partial}{\partial U} \left(\mathcal{V}_W{}^{\mathsf{T}} \lambda\right) \tag{34}$$
$$= - [\lambda] [\mathcal{V}]^{-3} [U] [W] \tag{35}$$

$$\mathcal{V}_{WW}(\lambda) = \frac{\partial}{\partial W} \left(\mathcal{V}_W{}^{\mathsf{T}} \lambda\right) \tag{36}$$
$$= [\lambda] [\mathcal{V}]^{-3} [U]^2 \tag{37}$$

3.2 Branch Voltages

The $n_l \times 1$ vectors of complex voltages at the *from* and *to* ends of all branches are, respectively

$$V_f = C_f V \tag{38}$$
$$V_t = C_t V \tag{39}$$

3.2.1 First Derivatives

$$\frac{\partial V_f}{\partial U} = C_f \frac{\partial V}{\partial U} = C_f \tag{40}$$

$$\frac{\partial V_f}{\partial W} = C_f \frac{\partial V}{\partial W} = jC_f \tag{41}$$

3.3 Reference Bus Voltage Angles

The $n_r \times 1$ vector of complex voltages at reference buses is

$$V_{\text{ref}} = C_{\text{ref}} V \tag{42}$$

The equality constraint on voltage angles at reference buses is

$$G^{\text{ref}}(X) = C_{\text{ref}}\Theta - \Theta_{\text{ref}}^{\text{specified}} \tag{43}$$

3.3.1 First Derivatives

$$G_U^{\text{ref}} = \frac{\partial G^{\text{ref}}}{\partial U} = C_{\text{ref}}\Theta_U = -C_{\text{ref}} \left[\mathcal{V}\right]^{-2} \left[W\right] \tag{44}$$

$$G_W^{\text{ref}} = \frac{\partial G^{\text{ref}}}{\partial W} = C_{\text{ref}}\Theta_W = C_{\text{ref}} \left[\mathcal{V}\right]^{-2} \left[U\right] \tag{45}$$

3.3.2 Second Derivatives

$$G_{UU}^{\text{ref}}(\lambda) = C_{\text{ref}}\Theta_{UU}(\lambda) = 2C_{\text{ref}} \left[\lambda\right] \left[\mathcal{V}\right]^{-4} \left[U\right] \left[W\right] \tag{46}$$

$$G_{UW}^{\text{ref}}(\lambda) = C_{\text{ref}}\Theta_{UW}(\lambda) = C_{\text{ref}} \left[\lambda\right] \left[\mathcal{V}\right]^{-4} \left(\left[W\right]^2 - \left[U\right]^2\right) \tag{47}$$

$$G_{WU}^{\text{ref}}(\lambda) = C_{\text{ref}}\Theta_{WU}(\lambda) = C_{\text{ref}} \left[\lambda\right] \left[\mathcal{V}\right]^{-4} \left(\left[W\right]^2 - \left[U\right]^2\right) \tag{48}$$

$$G_{WW}^{\text{ref}}(\lambda) = C_{\text{ref}}\Theta_{WW}(\lambda) = -2C_{\text{ref}} \left[\lambda\right] \left[\mathcal{V}\right]^{-4} \left[U\right] \left[W\right] \tag{49}$$

3.4 Bus Voltage Magnitude Limits

Upper and lower bounds on bus voltage magnitudes are the $n_b \times 1$ vectors

$$H^{\mathcal{V}^{\max}}(X) = \mathcal{V} - \mathcal{V}^{\max} \tag{50}$$

$$H^{\mathcal{V}^{\min}}(X) = \mathcal{V}^{\min} - \mathcal{V} \tag{51}$$

3.4.1 First Derivatives

$$H_U^{\mathcal{V}^{\max}} = \mathcal{V}_U = [\mathcal{V}]^{-1}[U] \tag{52}$$

$$H_W^{\mathcal{V}^{\max}} = \mathcal{V}_W = [\mathcal{V}]^{-1}[W] \tag{53}$$

$$H_U^{\mathcal{V}^{\min}} = -H_U^{\mathcal{V}^{\max}} \tag{54}$$

$$H_W^{\mathcal{V}^{\min}} = -H_W^{\mathcal{V}^{\max}} \tag{55}$$

3.4.2 Second Derivatives

$$H_{UU}^{\mathcal{V}^{\max}}(\lambda) = \mathcal{V}_{UU}(\lambda) = [\lambda][\mathcal{V}]^{-3}[W]^2 \tag{56}$$

$$H_{UW}^{\mathcal{V}^{\max}}(\lambda) = \mathcal{V}_{UW}(\lambda) = -[\lambda][\mathcal{V}]^{-3}[U][W] \tag{57}$$

$$H_{WU}^{\mathcal{V}^{\max}}(\lambda) = \mathcal{V}_{WU}(\lambda) = -[\lambda][\mathcal{V}]^{-3}[U][W] \tag{58}$$

$$H_{WW}^{\mathcal{V}^{\max}}(\lambda) = \mathcal{V}_{WW}(\lambda) = [\lambda][\mathcal{V}]^{-3}[U]^2 \tag{59}$$

$$H_{UU}^{\mathcal{V}^{\min}}(\lambda) = -H_{UU}^{\mathcal{V}^{\max}}(\lambda) \tag{60}$$

$$H_{UW}^{\mathcal{V}^{\min}}(\lambda) = -H_{UW}^{\mathcal{V}^{\max}}(\lambda) \tag{61}$$

$$H_{WU}^{\mathcal{V}^{\min}}(\lambda) = -H_{WU}^{\mathcal{V}^{\max}}(\lambda) \tag{62}$$

$$H_{WW}^{\mathcal{V}^{\min}}(\lambda) = -H_{WW}^{\mathcal{V}^{\max}}(\lambda) \tag{63}$$

3.5 Branch Angle Difference Limits

Upper and lower bounds on branch voltage angle differences are the $n_l \times 1$ vectors

$$H^{\Theta^{\max}}(X) = (C_f - C_t)\Theta - \Theta_{ft}^{\max} \tag{64}$$

$$H^{\Theta^{\min}}(X) = \Theta_{ft}^{\min} - (C_f - C_t)\Theta \tag{65}$$

3.5.1 First Derivatives

$$H_U^{\Theta^{\max}} = (C_f - C_t)\Theta_U = -(C_f - C_t)\left[\mathcal{V}\right]^{-2}\left[W\right] \tag{66}$$

$$H_W^{\Theta^{\max}} = \Theta_W = (C_f - C_t)\left[\mathcal{V}\right]^{-2}\left[U\right] \tag{67}$$

$$H_U^{\Theta^{\min}} = -H_U^{\Theta^{\max}} \tag{68}$$

$$H_W^{\Theta^{\min}} = -H_W^{\Theta^{\max}} \tag{69}$$

3.5.2 Second Derivatives

$$H_{UU}^{\Theta^{\max}}(\lambda) = (C_f - C_t)\Theta_{UU}(\lambda) = 2(C_f - C_t)\left[\lambda\right]\left[\mathcal{V}\right]^{-4}\left[U\right]\left[W\right] \tag{70}$$

$$H_{UW}^{\Theta^{\max}}(\lambda) = (C_f - C_t)\Theta_{UW}(\lambda) = (C_f - C_t)\left[\lambda\right]\left[\mathcal{V}\right]^{-4}\left(\left[W\right]^2 - \left[U\right]^2\right) \tag{71}$$

$$H_{WU}^{\Theta^{\max}}(\lambda) = (C_f - C_t)\Theta_{WU}(\lambda) = (C_f - C_t)\left[\lambda\right]\left[\mathcal{V}\right]^{-4}\left(\left[W\right]^2 - \left[U\right]^2\right) \tag{72}$$

$$H_{WW}^{\Theta^{\max}}(\lambda) = (C_f - C_t)\Theta_{WW}(\lambda) = -2(C_f - C_t)\left[\lambda\right]\left[\mathcal{V}\right]^{-4}\left[U\right]\left[W\right] \tag{73}$$

$$H_{UU}^{\Theta^{\min}}(\lambda) = -H_{UU}^{\Theta^{\max}}(\lambda) \tag{74}$$

$$H_{UW}^{\Theta^{\min}}(\lambda) = -H_{UW}^{\Theta^{\max}}(\lambda) \tag{75}$$

$$H_{WU}^{\Theta^{\min}}(\lambda) = -H_{WU}^{\Theta^{\max}}(\lambda) \tag{76}$$

$$H_{WW}^{\Theta^{\min}}(\lambda) = -H_{WW}^{\Theta^{\max}}(\lambda) \tag{77}$$

4 Bus Injections

4.1 Complex Current Injections

Consider the complex current balance equation, $G^c(X) = \mathbf{0}$, where

$$G^c(X) = I^{\text{bus}} + I^{dg} \tag{78}$$

and

$$I^{\text{bus}} = Y_{\text{bus}} V \tag{79}$$

$$I^{dg} = [S_d - C_g S_g]^* \Lambda^* \tag{80}$$

4.1.1 First Derivatives

$$I_X^{\text{bus}} = \frac{\partial I^{\text{bus}}}{\partial X} = \begin{bmatrix} I_U^{\text{bus}} & I_W^{\text{bus}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{81}$$

$$I_U^{\text{bus}} = \frac{\partial I^{\text{bus}}}{\partial U} = Y_{\text{bus}} \frac{\partial V}{\partial U} = Y_{\text{bus}} \tag{82}$$

$$I_W^{\text{bus}} = \frac{\partial I^{\text{bus}}}{\partial W} = Y_{\text{bus}} \frac{\partial V}{\partial W} = jY_{\text{bus}} \tag{83}$$

$$I_X^{dg} = \frac{\partial I^{dg}}{\partial X} = \begin{bmatrix} I_U^{dg} & I_W^{dg} & I_{P_g}^{dg} & I_{Q_g}^{dg} \end{bmatrix} \tag{84}$$

$$I_U^{dg} = \frac{\partial I^{dg}}{\partial U} = -[S_d - C_g S_g]^* \, [\Lambda^*]^2 \tag{85}$$

$$I_W^{dg} = \frac{\partial I^{dg}}{\partial W} = j[S_d - C_g S_g]^* \, [\Lambda^*]^2 \tag{86}$$

$$I_{P_g}^{dg} = \frac{\partial I^{dg}}{\partial P_g} = - \, [\Lambda^*] \, C_g \tag{87}$$

$$I_{Q_g}^{dg} = \frac{\partial I^{dg}}{\partial Q_g} = j \, [\Lambda^*] \, C_g \tag{88}$$

$$G^c_X = \frac{\partial G^c}{\partial X} = \left[\begin{array}{cccc} G^c_U & G^c_W & G^c_{P_g} & G^c_{Q_g} \end{array} \right] \tag{89}$$

$$G^c_U = \frac{\partial G^c}{\partial U} = I^{\text{bus}}_U + I^{dg}_U = Y_{\text{bus}} - [S_d - C_g S_g]^* \left[\Lambda^*\right]^2 \tag{90}$$

$$G^c_W = \frac{\partial G^c}{\partial W} = I^{\text{bus}}_W + I^{dg}_W = j \left(Y_{\text{bus}} + [S_d - C_g S_g]^* \left[\Lambda^*\right]^2 \right) \tag{91}$$

$$G^c_{P_g} = \frac{\partial G^c}{\partial P_g} = I^{dg}_{P_g} = - \left[\Lambda^*\right] C_g \tag{92}$$

$$G^c_{Q_g} = \frac{\partial G^c}{\partial Q_g} = I^{dg}_{Q_g} = j \left[\Lambda^*\right] C_g \tag{93}$$

4.1.2 Second Derivatives

$$I^{\text{bus}}_{XX}(\lambda) = \frac{\partial}{\partial X} \left({I^{\text{bus}}_X}^{\mathsf{T}} \lambda \right) = \mathbf{0} \tag{94}$$

$$I^{dg}_{XX}(\lambda) = \frac{\partial}{\partial X} \left({I^{dg}_X}^{\mathsf{T}} \lambda \right) \tag{95}$$

$$= \left[\begin{array}{cccc} I^{dg}_{UU}(\lambda) & I^{dg}_{UW}(\lambda) & I^{dg}_{UP_g}(\lambda) & I^{dg}_{UQ_g}(\lambda) \\ I^{dg}_{WU}(\lambda) & I^{dg}_{WW}(\lambda) & I^{dg}_{WP_g}(\lambda) & I^{dg}_{WQ_g}(\lambda) \\ I^{dg}_{P_gU}(\lambda) & I^{dg}_{P_gW}(\lambda) & \mathbf{0} & \mathbf{0} \\ I^{dg}_{Q_gU}(\lambda) & I^{dg}_{Q_gW}(\lambda) & \mathbf{0} & \mathbf{0} \end{array} \right] \tag{96}$$

$$= \left[\begin{array}{cccc} \mathcal{C} & -j\mathcal{C} & \mathcal{D}^{\mathsf{T}} & -j\mathcal{D}^{\mathsf{T}} \\ -j\mathcal{C} & -\mathcal{C} & -j\mathcal{D}^{\mathsf{T}} & -\mathcal{D}^{\mathsf{T}} \\ \mathcal{D} & -j\mathcal{D} & \mathbf{0} & \mathbf{0} \\ -j\mathcal{D} & -\mathcal{D} & \mathbf{0} & \mathbf{0} \end{array} \right] \tag{97}$$

$$I^{dg}_{UU}(\lambda) = \frac{\partial}{\partial U} \left({I^{dg}_U}^{\mathsf{T}} \lambda \right) \tag{98}$$

$$= \frac{\partial}{\partial U} \left(-[S_d - C_g S_g]^* \left[\Lambda^*\right]^2 \lambda \right) \tag{99}$$

$$
\begin{aligned}
&= -2[S_d - C_g S_g]^* \, [\lambda] \, [\Lambda^*] \, \Lambda_U^* && (100) \\
&= 2[S_d - C_g S_g]^* \, [\lambda] \, [\Lambda^*]^3 && (101) \\
&= \mathcal{C} && (102)
\end{aligned}
$$

$$
\begin{aligned}
I_{WU}^{dg}(\lambda) &= \frac{\partial}{\partial U} \left({I_W^{dg}}^\mathsf{T} \lambda \right) && (103) \\
&= \frac{\partial}{\partial U} \left(j[S_d - C_g S_g]^* \, [\Lambda^*]^2 \, \lambda \right) && (104) \\
&= 2j[S_d - C_g S_g]^* \, [\lambda] \, [\Lambda^*] \, \Lambda_U^* && (105) \\
&= -2j[S_d - C_g S_g]^* \, [\lambda] \, [\Lambda^*]^3 && (106) \\
&= -j\mathcal{C} && (107)
\end{aligned}
$$

$$
\begin{aligned}
I_{P_g U}^{dg}(\lambda) &= \frac{\partial}{\partial U} \left({I_{P_g}^{dg}}^\mathsf{T} \lambda \right) && (108) \\
&= \frac{\partial}{\partial U} \left(-{C_g}^\mathsf{T} \, [\Lambda^*] \, \lambda \right) && (109) \\
&= -{C_g}^\mathsf{T} \, [\lambda] \, \Lambda_U^* && (110) \\
&= {C_g}^\mathsf{T} \, [\lambda] \, [\Lambda^*]^2 && (111) \\
&= \mathcal{D} && (112)
\end{aligned}
$$

$$
\begin{aligned}
I_{Q_g U}^{dg}(\lambda) &= \frac{\partial}{\partial U} \left({I_{Q_g}^{dg}}^\mathsf{T} \lambda \right) && (113) \\
&= \frac{\partial}{\partial U} \left(j{C_g}^\mathsf{T} \, [\Lambda^*] \, \lambda \right) && (114) \\
&= j{C_g}^\mathsf{T} \, [\lambda] \, \Lambda_U^* && (115) \\
&= -j{C_g}^\mathsf{T} \, [\lambda] \, [\Lambda^*]^2 && (116) \\
&= -j\mathcal{D} && (117)
\end{aligned}
$$

$$
\begin{aligned}
I_{UW}^{dg}(\lambda) &= \frac{\partial}{\partial W} \left({I_U^{dg}}^\mathsf{T} \lambda \right) && (118) \\
&= \frac{\partial}{\partial W} \left(-[S_d - C_g S_g]^* \, [\Lambda^*]^2 \, \lambda \right) && (119) \\
&= -2[S_d - C_g S_g]^* \, [\lambda] \, [\Lambda^*] \, \Lambda_W^* && (120)
\end{aligned}
$$

$$= -2j[S_d - C_g S_g]^* [\lambda] [\Lambda^*]^3 \tag{121}$$
$$= {I^{dg}_{WU}}^{\mathsf{T}}(\lambda) = -j\mathcal{C} \tag{122}$$

$$I^{dg}_{WW}(\lambda) = \frac{\partial}{\partial W}\left({I^{dg}_{W}}^{\mathsf{T}}\lambda\right) \tag{123}$$
$$= \frac{\partial}{\partial W}\left(j[S_d - C_g S_g]^* [\Lambda^*]^2 \lambda\right) \tag{124}$$
$$= 2j[S_d - C_g S_g]^* [\lambda] [\Lambda^*] \Lambda^*_W \tag{125}$$
$$= -2[S_d - C_g S_g]^* [\lambda] [\Lambda^*]^3 \tag{126}$$
$$= -\mathcal{C} \tag{127}$$

$$I^{dg}_{P_g W}(\lambda) = \frac{\partial}{\partial W}\left({I^{dg}_{P_g}}^{\mathsf{T}}\lambda\right) \tag{128}$$
$$= \frac{\partial}{\partial W}\left(-{C_g}^{\mathsf{T}} [\Lambda^*] \lambda\right) \tag{129}$$
$$= -{C_g}^{\mathsf{T}} [\lambda] \Lambda^*_W \tag{130}$$
$$= -j{C_g}^{\mathsf{T}} [\lambda] [\Lambda^*]^2 \tag{131}$$
$$= -j\mathcal{D} \tag{132}$$

$$I^{dg}_{Q_g W}(\lambda) = \frac{\partial}{\partial W}\left({I^{dg}_{Q_g}}^{\mathsf{T}}\lambda\right) \tag{133}$$
$$= \frac{\partial}{\partial W}\left(j{C_g}^{\mathsf{T}} [\Lambda^*] \lambda\right) \tag{134}$$
$$= j{C_g}^{\mathsf{T}} [\lambda] \Lambda^*_W \tag{135}$$
$$= -{C_g}^{\mathsf{T}} [\lambda] [\Lambda^*]^2 \tag{136}$$
$$= -\mathcal{D} \tag{137}$$

$$I^{dg}_{U P_g}(\lambda) = \frac{\partial}{\partial P_g}\left({I^{dg}_{U}}^{\mathsf{T}}\lambda\right) \tag{138}$$
$$= \frac{\partial}{\partial P_g}\left(-[S_d - C_g S_g]^* [\Lambda^*]^2 \lambda\right) \tag{139}$$
$$= [\Lambda^*]^2 [\lambda] C_g \tag{140}$$

$$= {I^{dg}_{P_g U}}^{\mathsf{T}}(\lambda) = \mathcal{D}^{\mathsf{T}} \tag{141}$$

$$I^{dg}_{WP_g}(\lambda) = \frac{\partial}{\partial P_g}\left({I^{dg}_W}^{\mathsf{T}}\lambda\right) \tag{142}$$

$$= \frac{\partial}{\partial P_g}\left(j[S_d - C_g S_g]^* \left[\Lambda^*\right]^2 \lambda\right) \tag{143}$$

$$= -j\left[\Lambda^*\right]^2 [\lambda] \, C_g \tag{144}$$

$$= {I^{dg}_{P_g W}}^{\mathsf{T}}(\lambda) = -j\mathcal{D}^{\mathsf{T}} \tag{145}$$

$$I^{dg}_{UQ_g}(\lambda) = \frac{\partial}{\partial Q_g}\left({I^{dg}_U}^{\mathsf{T}}\lambda\right) \tag{146}$$

$$= \frac{\partial}{\partial Q_g}\left(-[S_d - C_g S_g]^* \left[\Lambda^*\right]^2 \lambda\right) \tag{147}$$

$$= -j\left[\Lambda^*\right]^2 [\lambda] \, C_g \tag{148}$$

$$= {I^{dg}_{Q_g U}}^{\mathsf{T}}(\lambda) = -j\mathcal{D}^{\mathsf{T}} \tag{149}$$

$$I^{dg}_{WQ_g}(\lambda) = \frac{\partial}{\partial Q_g}\left({I^{dg}_W}^{\mathsf{T}}\lambda\right) \tag{150}$$

$$= \frac{\partial}{\partial Q_g}\left(j[S_d - C_g S_g]^* \left[\Lambda^*\right]^2 \lambda\right) \tag{151}$$

$$= -\left[\Lambda^*\right]^2 [\lambda] \, C_g \tag{152}$$

$$= {I^{dg}_{Q_g W}}^{\mathsf{T}}(\lambda) = -\mathcal{D}^{\mathsf{T}} \tag{153}$$

$$G^c_{XX}(\lambda) = \frac{\partial}{\partial X}\left({G^c_X}^{\mathsf{T}}\lambda\right) \tag{154}$$

$$= \begin{bmatrix} G^c_{UU}(\lambda) & G^c_{UW}(\lambda) & G^c_{UP_g}(\lambda) & G^c_{UQ_g}(\lambda) \\ G^c_{WU}(\lambda) & G^c_{WW}(\lambda) & G^c_{WP_g}(\lambda) & G^c_{WQ_g}(\lambda) \\ G^c_{P_g U}(\lambda) & G^c_{P_g W}(\lambda) & \mathbf{0} & \mathbf{0} \\ G^c_{Q_g U}(\lambda) & G^c_{Q_g W}(\lambda) & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{155}$$

$$= I^{dg}_{XX}(\lambda) \tag{156}$$

$$= \begin{bmatrix} \mathcal{C} & -j\mathcal{C} & \mathcal{D}^\mathsf{T} & -j\mathcal{D}^\mathsf{T} \\ -j\mathcal{C} & -\mathcal{C} & -j\mathcal{D}^\mathsf{T} & -\mathcal{D}^\mathsf{T} \\ \mathcal{D} & -j\mathcal{D} & \mathbf{0} & \mathbf{0} \\ -j\mathcal{D} & -\mathcal{D} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{157}$$

Computational savings can be achieved by storing and reusing certain intermediate terms during the computation of these second derivatives, as follows:

$$\mathcal{A} = [\Lambda^*] \tag{158}$$
$$\mathcal{B} = [\lambda]\,\mathcal{A}^2 \tag{159}$$
$$\mathcal{C} = 2[S_d - C_g S_g]^* \mathcal{A}\mathcal{B} \tag{160}$$
$$\mathcal{D} = {C_g}^\mathsf{T} \mathcal{B} \tag{161}$$
$$G^c_{UU}(\lambda) = \mathcal{C} \tag{162}$$
$$G^c_{WU}(\lambda) = -j\mathcal{C} \tag{163}$$
$$G^c_{P_gU}(\lambda) = \mathcal{D} \tag{164}$$
$$G^c_{Q_gU}(\lambda) = -j\mathcal{D} \tag{165}$$
$$G^c_{WW}(\lambda) = -\mathcal{C} \tag{166}$$
$$G^c_{P_gW}(\lambda) = -j\mathcal{D} \tag{167}$$
$$G^c_{Q_gW}(\lambda) = -\mathcal{D} \tag{168}$$
$$G^c_{UW}(\lambda) = G^c_{WU}(\lambda) \tag{169}$$
$$G^c_{UP_g}(\lambda) = {G^c_{P_gU}}^\mathsf{T}(\lambda) \tag{170}$$
$$G^c_{WP_g}(\lambda) = {G^c_{P_gW}}^\mathsf{T}(\lambda) \tag{171}$$
$$G^c_{UQ_g}(\lambda) = {G^c_{Q_gU}}^\mathsf{T}(\lambda) \tag{172}$$
$$G^c_{WQ_g}(\lambda) = {G^c_{Q_gW}}^\mathsf{T}(\lambda) \tag{173}$$

4.2 Complex Power Injections

Consider the complex power balance equation, $G^s(X) = \mathbf{0}$, where

$$G^s(X) = S^{\text{bus}} + S_d - C_g S_g \tag{174}$$

and

$$S^{\text{bus}} = [V]\, I^{\text{bus}*} \tag{175}$$

4.2.1 First Derivatives

$$G_X^s = \frac{\partial G^s}{\partial X} = \left[\begin{array}{cccc} G_U^s & G_W^s & G_{P_g}^s & G_{Q_g}^s \end{array}\right] \tag{176}$$

$$G_U^s = \frac{\partial S^{\text{bus}}}{\partial U} = \left[I^{\text{bus}*}\right] \frac{\partial V}{\partial U} + [V] \frac{\partial I^{\text{bus}*}}{\partial U} \tag{177}$$

$$= \left[I^{\text{bus}*}\right] + [V] \, Y_{\text{bus}}{}^{*} \tag{178}$$

$$G_W^s = \frac{\partial S^{\text{bus}}}{\partial W} = \left[I^{\text{bus}*}\right] \frac{\partial V}{\partial W} + [V] \frac{\partial I^{\text{bus}*}}{\partial W} \tag{179}$$

$$= j\left(\left[I^{\text{bus}*}\right] - [V] \, Y_{\text{bus}}{}^{*}\right) \tag{180}$$

$$G_{P_g}^s = -C_g \tag{181}$$

$$G_{Q_g}^s = -jC_g \tag{182}$$

4.2.2 Second Derivatives

$$G_{XX}^s(\lambda) = \frac{\partial}{\partial X}\left({G_X^s}^{\mathsf{T}} \lambda\right) \tag{183}$$

$$= \begin{bmatrix} G_{UU}^s(\lambda) & G_{UW}^s(\lambda) & \mathbf{0} & \mathbf{0} \\ G_{WU}^s(\lambda) & G_{WW}^s(\lambda) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{184}$$

$$G_{UU}^s(\lambda) = \frac{\partial}{\partial U}\left({G_U^s}^{\mathsf{T}} \lambda\right) \tag{185}$$

$$= \frac{\partial}{\partial U}\left(\left(\left[I^{\text{bus}*}\right] + Y_{\text{bus}}{}^{*\mathsf{T}} [V]\right) \lambda\right) \tag{186}$$

$$= \frac{\partial}{\partial U}\left([\lambda] \, Y_{\text{bus}}{}^{*} V^{*} + Y_{\text{bus}}{}^{*\mathsf{T}} [\lambda] \, V\right) \tag{187}$$

$$= [\lambda] \, Y_{\text{bus}}{}^{*} + Y_{\text{bus}}{}^{*\mathsf{T}} [\lambda] \tag{188}$$

$$= \mathcal{F} \tag{189}$$

$$G_{WU}^s(\lambda) = \frac{\partial}{\partial U}\left({G_W^s}^{\mathsf{T}} \lambda\right) \tag{190}$$

$$= \frac{\partial}{\partial U}\left(j\left(\left[I^{\text{bus}*}\right] - {Y_{\text{bus}}}^{*\mathsf{T}}[V]\right)\lambda\right) \tag{191}$$

$$= \frac{\partial}{\partial U}\left(j\left([\lambda]\,{Y_{\text{bus}}}^{*}V^{*} - {Y_{\text{bus}}}^{*\mathsf{T}}[\lambda]\,V\right)\right) \tag{192}$$

$$= j\left([\lambda]\,{Y_{\text{bus}}}^{*} - {Y_{\text{bus}}}^{*\mathsf{T}}[\lambda]\right) \tag{193}$$

$$= \mathcal{G} \tag{194}$$

$$G^s_{UW}(\lambda) = \frac{\partial}{\partial W}\left({G^s_U}^{\mathsf{T}}\lambda\right) \tag{195}$$

$$= \frac{\partial}{\partial W}\left(\left(\left[I^{\text{bus}*}\right] + {Y_{\text{bus}}}^{*\mathsf{T}}[V]\right)\lambda\right) \tag{196}$$

$$= \frac{\partial}{\partial W}\left([\lambda]\,{Y_{\text{bus}}}^{*}V^{*} + {Y_{\text{bus}}}^{*\mathsf{T}}[\lambda]\,V\right) \tag{197}$$

$$= j\left({Y_{\text{bus}}}^{*\mathsf{T}}[\lambda] - [\lambda]\,{Y_{\text{bus}}}^{*}\right) \tag{198}$$

$$= {G^s_{WU}}^{\mathsf{T}}(\lambda) = \mathcal{G}^{\mathsf{T}} \tag{199}$$

$$G^s_{WW}(\lambda) = \frac{\partial}{\partial W}\left({G^s_W}^{\mathsf{T}}\lambda\right) \tag{200}$$

$$= \frac{\partial}{\partial W}\left(j\left(\left[I^{\text{bus}*}\right] - {Y_{\text{bus}}}^{*\mathsf{T}}[V]\right)\lambda\right) \tag{201}$$

$$= \frac{\partial}{\partial W}\left(j\left([\lambda]\,{Y_{\text{bus}}}^{*}V^{*} - {Y_{\text{bus}}}^{*\mathsf{T}}[\lambda]\,V\right)\right) \tag{202}$$

$$= [\lambda]\,{Y_{\text{bus}}}^{*} + {Y_{\text{bus}}}^{*\mathsf{T}}[\lambda] \tag{203}$$

$$= \mathcal{F} \tag{204}$$

Computational savings can be achieved by storing and reusing certain intermediate terms during the computation of these second derivatives, as follows:

$$\mathcal{E} = [\lambda]\,{Y_{\text{bus}}}^{*} \tag{205}$$

$$\mathcal{F} = \mathcal{E} + \mathcal{E}^{\mathsf{T}} \tag{206}$$

$$\mathcal{G} = j\left(\mathcal{E} - \mathcal{E}^{\mathsf{T}}\right) \tag{207}$$

$$G^s_{UU}(\lambda) = \mathcal{F} \tag{208}$$

$$G^s_{WU}(\lambda) = \mathcal{G} \tag{209}$$

$$G^s_{UW}(\lambda) = \mathcal{G}^{\mathsf{T}} \tag{210}$$

$$G^s_{WW}(\lambda) = \mathcal{F} \tag{211}$$

5 Branch Flows

Consider the line flow constraints of the form $H(X) < \mathbf{0}$. This section examines 3 variations based on the square of the magnitude of the current, apparent power and real power, respectively. The relationships are derived first for the complex flows at the *from* ends of the branches. Derivations for the *to* end are identical (i.e. just replace all f sub/super-scripts with t).

5.1 Complex Currents

$$I^f = Y_f V \tag{212}$$
$$I^t = Y_t V \tag{213}$$

5.1.1 First Derivatives

$$I_X^f = \frac{\partial I^f}{\partial X} = \begin{bmatrix} I_U^f & I_W^f & I_{P_g}^f & I_{Q_g}^f \end{bmatrix} \tag{214}$$

$$I_U^f = \frac{\partial I^f}{\partial U} = Y_f \tag{215}$$
$$I_W^f = \frac{\partial I^f}{\partial W} = jY_f \tag{216}$$
$$I_{P_g}^f = \frac{\partial I^f}{\partial P_g} = \mathbf{0} \tag{217}$$
$$I_{Q_g}^f = \frac{\partial I^f}{\partial Q_g} = \mathbf{0} \tag{218}$$

5.1.2 Second Derivatives

$$I_{XX}^f(\mu) = \frac{\partial}{\partial X}\left({I_X^f}^\mathsf{T} \mu \right) = \mathbf{0} \tag{219}$$

5.2 Complex Power Flows

$$S^f = [V_f]\, {I^f}^* \tag{220}$$
$$S^t = [V_t]\, {I^t}^* \tag{221}$$

5.2.1 First Derivatives

$$S_X^f = \frac{\partial S^f}{\partial X} = \left[\begin{array}{cccc} S_U^f & S_W^f & S_{P_g}^f & S_{Q_g}^f \end{array}\right] \tag{222}$$

$$= \left[I^{f^*}\right] \frac{\partial V_f}{\partial X} + [V_f] \frac{\partial I^{f^*}}{\partial X} \tag{223}$$

$$S_U^f = \left[I^{f^*}\right] \frac{\partial V_f}{\partial U} + [V_f] \frac{\partial I^{f^*}}{\partial U} \tag{224}$$

$$= \left[I^{f^*}\right] C_f + [V_f] {Y_f}^* \tag{225}$$

$$S_W^f = \left[I^{f^*}\right] \frac{\partial V_f}{\partial W} + [V_f] \frac{\partial I^{f^*}}{\partial W} \tag{226}$$

$$= j\left(\left[I^{f^*}\right] C_f - [V_f] {Y_f}^*\right) \tag{227}$$

$$S_{P_g}^f = \mathbf{0} \tag{228}$$

$$S_{Q_g}^f = \mathbf{0} \tag{229}$$

5.2.2 Second Derivatives

$$S_{XX}^f(\mu) = \frac{\partial}{\partial X}\left({S_X^f}^{\mathsf{T}} \mu\right) \tag{230}$$

$$= \left[\begin{array}{cccc} S_{UU}^f(\mu) & S_{UW}^f(\mu) & \mathbf{0} & \mathbf{0} \\ S_{WU}^f(\mu) & S_{WW}^f(\mu) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \tag{231}$$

$$S_{UU}^f(\mu) = \frac{\partial}{\partial U}\left({S_U^f}^{\mathsf{T}} \mu\right) \tag{232}$$

$$= \frac{\partial}{\partial U}\left(\left({C_f}^{\mathsf{T}} \left[I^{f^*}\right] + {Y_f}^{*\mathsf{T}} [V_f]\right) \mu\right) \tag{233}$$

$$= {C_f}^{\mathsf{T}} [\mu] \frac{\partial I^{f^*}}{\partial U} + {Y_f}^{*\mathsf{T}} [\mu] \frac{\partial V_f}{\partial U} \tag{234}$$

$$= {C_f}^{\mathsf{T}} [\mu] {Y_f}^* + {Y_f}^{*\mathsf{T}} [\mu] C_f \tag{235}$$

$$= \mathcal{B}_f \tag{236}$$

$$
\begin{aligned}
S^f_{WU}(\mu) &= \frac{\partial}{\partial U}\left({S^f_W}^\mathsf{T}\mu\right) &(237)\\
&= \frac{\partial}{\partial U}\left(j\left({C_f}^\mathsf{T}\left[I^{f*}\right] - {Y_f^*}^\mathsf{T}\left[V_f\right]\right)\mu\right) &(238)\\
&= j\left({C_f}^\mathsf{T}\left[\mu\right]\frac{\partial I^{f*}}{\partial U} - {Y_f^*}^\mathsf{T}\left[\mu\right]\frac{\partial V_f}{\partial U}\right) &(239)\\
&= j\left({C_f}^\mathsf{T}\left[\mu\right]Y_f^* - {Y_f^*}^\mathsf{T}\left[\mu\right]C_f\right) &(240)\\
&= \mathcal{D}_f &(241)
\end{aligned}
$$

$$
\begin{aligned}
S^f_{UW}(\mu) &= \frac{\partial}{\partial W}\left({S^f_U}^\mathsf{T}\mu\right) &(242)\\
&= \frac{\partial}{\partial W}\left(\left({C_f}^\mathsf{T}\left[I^{f*}\right] + {Y_f^*}^\mathsf{T}\left[V_f\right]\right)\mu\right) &(243)\\
&= {C_f}^\mathsf{T}\left[\mu\right]\frac{\partial I^{f*}}{\partial W} + {Y_f^*}^\mathsf{T}\left[\mu\right]\frac{\partial V_f}{\partial W} &(244)\\
&= -j\left({C_f}^\mathsf{T}\left[\mu\right]Y_f^* - {Y_f^*}^\mathsf{T}\left[\mu\right]C_f\right) &(245)\\
&= {S^f_{WU}}^\mathsf{T}(\mu) = -\mathcal{D}_f &(246)
\end{aligned}
$$

$$
\begin{aligned}
S^f_{WW}(\mu) &= \frac{\partial}{\partial W}\left({S^f_W}^\mathsf{T}\mu\right) &(247)\\
&= \frac{\partial}{\partial W}\left(j\left({C_f}^\mathsf{T}\left[I^{f*}\right] - {Y_f^*}^\mathsf{T}\left[V_f\right]\right)\mu\right) &(248)\\
&= j\left({C_f}^\mathsf{T}\left[\mu\right]\frac{\partial I^{f*}}{\partial W} - {Y_f^*}^\mathsf{T}\left[\mu\right]\frac{\partial V_f}{\partial W}\right) &(249)\\
&= j\left({C_f}^\mathsf{T}\left[\mu\right](-j)Y_f^* - {Y_f^*}^\mathsf{T}\left[\mu\right](j)C_f\right) &(250)\\
&= {C_f}^\mathsf{T}\left[\mu\right]Y_f^* + {Y_f^*}^\mathsf{T}\left[\mu\right]C_f &(251)\\
&= \mathcal{B}_f &(252)
\end{aligned}
$$

Computational savings can be achieved by storing and reusing certain intermediate terms during the computation of these second derivatives, as follows:

$$\mathcal{A}_f = {C_f}^\mathsf{T}\left[\mu\right]Y_f^* \tag{253}$$

$$\mathcal{B}_f = \mathcal{A}_f + {\mathcal{A}_f}^\mathsf{T} \tag{254}$$
$$\mathcal{D}_f = j\left(\mathcal{A}_f - {\mathcal{A}_f}^\mathsf{T}\right) \tag{255}$$
$$S^f_{UU}(\mu) = \mathcal{B}_f \tag{256}$$
$$S^f_{WU}(\mu) = \mathcal{D}_f \tag{257}$$
$$S^f_{UW}(\mu) = {S^f_{WU}}^\mathsf{T}(\mu) = -\mathcal{D}_f \tag{258}$$
$$S^f_{WW}(\mu) = \mathcal{B}_f \tag{259}$$

5.3 Squared Current Magnitudes

See the corresponding section in MATPOWER *Technical Note 2*.

5.4 Squared Apparent Power Magnitudes

See the corresponding section in MATPOWER *Technical Note 2*.

5.5 Squared Real Power Magnitudes

See the corresponding section in MATPOWER *Technical Note 2*.

6 Generalized AC OPF Costs

Let X be defined as

$$X = \begin{bmatrix} U \\ W \\ P_g \\ Q_g \\ Y \\ Z \end{bmatrix} \tag{260}$$

where Y is the $n_y \times 1$ vector of cost variables associated with piecewise linear generator costs and Z is an $n_z \times 1$ vector of additional linearly constrained user variables.

See the corresponding section in MATPOWER *Technical Note 2* for additional details.

7 Lagrangian of the AC OPF

Consider the following AC OPF problem formulation, where X is defined as in (260), f is the cost function, and $\mathcal{X}$ represents the reduced form of X, consisting of only U, W, P_g and Q_g, without Y and Z.

$$\min_{X} f(X) \tag{261}$$

subject to

$$G(X) = \mathbf{0} \tag{262}$$

$$H(X) \leq \mathbf{0} \tag{263}$$

where

$$G(X) = \begin{bmatrix} \Re\{G^b(\mathcal{X})\} \\ \Im\{G^b(\mathcal{X})\} \\ G^{\text{ref}}(\mathcal{X}) \\ A_E X - B_E \end{bmatrix} \tag{264}$$

and

$$H(X) = \begin{bmatrix} H^f(\mathcal{X}) \\ H^t(\mathcal{X}) \\ H^{\mathcal{V}^{\max}}(\mathcal{X}) \\ H^{\mathcal{V}^{\min}}(\mathcal{X}) \\ H^{\Theta^{\max}}(\mathcal{X}) \\ H^{\Theta^{\min}}(\mathcal{X}) \\ A_I X - B_I \end{bmatrix} \tag{265}$$

and G^b is the nodal balance function, equal to either G^c for current balance or to G^s for power balance.

Partitioning the corresponding multipliers λ and μ similarly,

$$\lambda = \begin{bmatrix} \lambda_P \\ \lambda_Q \\ \lambda_{\text{ref}} \\ \lambda_E \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_f \\ \mu_t \\ \mu_{\mathcal{V}^{\max}} \\ \mu_{\mathcal{V}^{\min}} \\ \mu_{\Theta^{\max}} \\ \mu_{\Theta^{\min}} \\ \mu_I \end{bmatrix} \tag{266}$$

the Lagrangian for this problem can be written as

$$\mathcal{L}(X, \lambda, \mu) = f(X) + \lambda^{\mathsf{T}} G(X) + \mu^{\mathsf{T}} H(X) \tag{267}$$

7.1 Nodal Current Balance

Let the nodal balance function G^b be the nodal complex current balance G^c.

7.1.1 First Derivatives

$$\mathcal{L}_X(X,\lambda,\mu) = f_X + \lambda^\mathsf{T} G_X + \mu^\mathsf{T} H_X \tag{268}$$
$$\mathcal{L}_\lambda(X,\lambda,\mu) = G^\mathsf{T}(X) \tag{269}$$
$$\mathcal{L}_\mu(X,\lambda,\mu) = H^\mathsf{T}(X) \tag{270}$$

where

$$G_X = \begin{bmatrix} \Re\{G^c_\mathcal{X}\} & \mathbf{0} & \mathbf{0} \\ \Im\{G^c_\mathcal{X}\} & \mathbf{0} & \mathbf{0} \\ G^{\mathrm{ref}}_\mathcal{X} & \mathbf{0} & \mathbf{0} \\ & A_E & \end{bmatrix} = \begin{bmatrix} \Re\{G^c_U\} & \Re\{G^c_W\} & \Re\{G^c_{P_g}\} & \Re\{G^c_{Q_g}\} & \mathbf{0} & \mathbf{0} \\ \Im\{G^c_U\} & \Im\{G^c_W\} & \Im\{G^c_{P_g}\} & \Im\{G^c_{Q_g}\} & \mathbf{0} & \mathbf{0} \\ G^{\mathrm{ref}}_U & G^{\mathrm{ref}}_W & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & A_E & & & \end{bmatrix} \tag{271}$$

and

$$H_X = \begin{bmatrix} H^f_\mathcal{X} & \mathbf{0} & \mathbf{0} \\ H^t_\mathcal{X} & \mathbf{0} & \mathbf{0} \\ H^{\mathcal{V}^{\max}}_\mathcal{X} & \mathbf{0} & \mathbf{0} \\ H^{\mathcal{V}^{\min}}_\mathcal{X} & \mathbf{0} & \mathbf{0} \\ H^{\Theta^{\max}}_\mathcal{X} & \mathbf{0} & \mathbf{0} \\ H^{\Theta^{\min}}_\mathcal{X} & \mathbf{0} & \mathbf{0} \\ & A_I & \end{bmatrix} = \begin{bmatrix} H^f_U & H^f_W & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ H^t_U & H^t_W & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ H^{\mathcal{V}^{\max}}_U & H^{\mathcal{V}^{\max}}_W & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ H^{\mathcal{V}^{\min}}_U & H^{\mathcal{V}^{\min}}_W & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ H^{\Theta^{\max}}_U & H^{\Theta^{\max}}_W & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ H^{\Theta^{\min}}_U & H^{\Theta^{\min}}_W & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & A_I & & & \end{bmatrix} \tag{272}$$

7.1.2 Second Derivatives

$$\mathcal{L}_{XX}(X,\lambda,\mu) = f_{XX} + G_{XX}(\lambda) + H_{XX}(\mu) \tag{273}$$

where

$$G_{XX}(\lambda) = \begin{bmatrix} G_{\mathcal{X}\mathcal{X}}(\lambda) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{274}$$

$$G_{\mathcal{X}\mathcal{X}}(\lambda) = \Re\{G^c_{\mathcal{X}\mathcal{X}}(\lambda_P)\} + \Im\{G^c_{\mathcal{X}\mathcal{X}}(\lambda_Q)\} + G^{\mathrm{ref}}_{\mathcal{X}\mathcal{X}}(\lambda_{\mathrm{ref}}) \tag{275}$$

$$\begin{aligned}
&= \Re\left\{\begin{bmatrix} G^c_{UU}(\lambda_P) & G^c_{UW}(\lambda_P) & G^c_{UP_g}(\lambda_P) & G^c_{UQ_g}(\lambda_P) \\ G^c_{WU}(\lambda_P) & G^c_{WW}(\lambda_P) & G^c_{WP_g}(\lambda_P) & G^c_{WQ_g}(\lambda_P) \\ G^c_{P_gU}(\lambda_P) & G^c_{P_gW}(\lambda_P) & \mathbf{0} & \mathbf{0} \\ G^c_{Q_gU}(\lambda_P) & G^c_{Q_gW}(\lambda_P) & \mathbf{0} & \mathbf{0} \end{bmatrix}\right\} \\
&+ \Im\left\{\begin{bmatrix} G^c_{UU}(\lambda_Q) & G^c_{UW}(\lambda_Q) & G^c_{UP_g}(\lambda_Q) & G^c_{UQ_g}(\lambda_Q) \\ G^c_{WU}(\lambda_Q) & G^c_{WW}(\lambda_Q) & G^c_{WP_g}(\lambda_Q) & G^c_{WQ_g}(\lambda_Q) \\ G^c_{P_gU}(\lambda_Q) & G^c_{P_gW}(\lambda_Q) & \mathbf{0} & \mathbf{0} \\ G^c_{Q_gU}(\lambda_Q) & G^c_{Q_gW}(\lambda_Q) & \mathbf{0} & \mathbf{0} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} G^{\mathrm{ref}}_{UU}(\lambda_{\mathrm{ref}}) & G^{\mathrm{ref}}_{UW}(\lambda_{\mathrm{ref}}) & \mathbf{0} & \mathbf{0} \\ G^{\mathrm{ref}}_{WU}(\lambda_{\mathrm{ref}}) & G^{\mathrm{ref}}_{WW}(\lambda_{\mathrm{ref}}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
\end{aligned} \tag{276}$$

and

$$H_{XX}(\mu) = \begin{bmatrix} H_{\mathcal{X}\mathcal{X}}(\mu) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{277}$$

$$H_{\mathcal{X}\mathcal{X}}(\mu) = \begin{bmatrix} H_{UU}(\mu) & H_{UW}(\mu) & \mathbf{0} & \mathbf{0} \\ H_{WU}(\mu) & H_{WW}(\mu) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{278}$$

$$\begin{aligned}
H_{UU}(\mu) &= H^f_{UU}(\mu_f) + H^t_{UU}(\mu_t) \\
&\quad + H^{\mathcal{V}^{\max}}_{UU}(\mu_{\mathcal{V}^{\max}}) + H^{\mathcal{V}^{\min}}_{UU}(\mu_{\mathcal{V}^{\min}}) \\
&\quad + H^{\Theta^{\max}}_{UU}(\mu_{\Theta^{\max}}) + H^{\Theta^{\min}}_{UU}(\mu_{\Theta^{\min}})
\end{aligned} \tag{279}$$

$$\begin{aligned}
H_{UW}(\mu) &= H^f_{UW}(\mu_f) + H^t_{UW}(\mu_t) \\
&\quad + H^{\mathcal{V}^{\max}}_{UW}(\mu_{\mathcal{V}^{\max}}) + H^{\mathcal{V}^{\min}}_{UW}(\mu_{\mathcal{V}^{\min}}) \\
&\quad + H^{\Theta^{\max}}_{UW}(\mu_{\Theta^{\max}}) + H^{\Theta^{\min}}_{UW}(\mu_{\Theta^{\min}})
\end{aligned} \tag{280}$$

$$\begin{aligned}
H_{WU}(\mu) &= H^f_{WU}(\mu_f) + H^t_{WU}(\mu_t) \\
&\quad + H^{\mathcal{V}^{\max}}_{WU}(\mu_{\mathcal{V}^{\max}}) + H^{\mathcal{V}^{\min}}_{WU}(\mu_{\mathcal{V}^{\min}})
\end{aligned}$$

$$+ H_{WU}^{\Theta^{\max}}(\mu_{\Theta^{\max}}) + H_{WU}^{\Theta^{\min}}(\mu_{\Theta^{\min}}) \tag{281}$$

$$\begin{aligned} H_{WW}(\mu) = & H_{WW}^{f}(\mu_f) + H_{WW}^{t}(\mu_t) \\ & + H_{WW}^{\mathcal{V}^{\max}}(\mu_{\mathcal{V}^{\max}}) + H_{WW}^{\mathcal{V}^{\min}}(\mu_{\mathcal{V}^{\min}}) \\ & + H_{WW}^{\Theta^{\max}}(\mu_{\Theta^{\max}}) + H_{WW}^{\Theta^{\min}}(\mu_{\Theta^{\min}}) \end{aligned} \tag{282}$$

7.2 Nodal Power Balance

Let the nodal balance function G^b be the nodal complex power balance G^s.

7.2.1 First Derivatives

$$\mathcal{L}_X(X, \lambda, \mu) = f_X + \lambda^{\mathsf{T}} G_X + \mu^{\mathsf{T}} H_X \tag{283}$$

$$\mathcal{L}_\lambda(X, \lambda, \mu) = G^{\mathsf{T}}(X) \tag{284}$$

$$\mathcal{L}_\mu(X, \lambda, \mu) = H^{\mathsf{T}}(X) \tag{285}$$

where

$$G_X = \begin{bmatrix} \Re\{G_{\mathcal{X}}^s\} & \mathbf{0} & \mathbf{0} \\ \Im\{G_{\mathcal{X}}^s\} & \mathbf{0} & \mathbf{0} \\ G_{\mathcal{X}}^{\text{ref}} & \mathbf{0} & \mathbf{0} \\ & A_E & \end{bmatrix} = \begin{bmatrix} \Re\{G_U^s\} & \Re\{G_W^s\} & -C_g & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Im\{G_U^s\} & \Im\{G_W^s\} & \mathbf{0} & -C_g & \mathbf{0} & \mathbf{0} \\ G_U^{\text{ref}} & G_W^{\text{ref}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & A_E & & & \end{bmatrix} \tag{286}$$

and H_X is the same as for nodal current balance in (272).

7.2.2 Second Derivatives

$$\mathcal{L}_{XX}(X, \lambda, \mu) = f_{XX} + G_{XX}(\lambda) + H_{XX}(\mu) \tag{287}$$

where

$$G_{XX}(\lambda) = \begin{bmatrix} G_{\mathcal{X}\mathcal{X}}(\lambda) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{288}$$

$$G_{\mathcal{X}\mathcal{X}}(\lambda) = \Re\{G^s_{\mathcal{X}\mathcal{X}}(\lambda_P)\} + \Im\{G^s_{\mathcal{X}\mathcal{X}}(\lambda_Q)\} + G^{\text{ref}}_{\mathcal{X}\mathcal{X}}(\lambda_{\text{ref}}) \tag{289}$$

$$\begin{aligned}
&= \Re\left\{\begin{bmatrix} G^s_{UU}(\lambda_P) & G^s_{UW}(\lambda_P) & \mathbf{0} & \mathbf{0} \\ G^s_{WU}(\lambda_P) & G^s_{WW}(\lambda_P) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\right\} \\
&\quad + \Im\left\{\begin{bmatrix} G^s_{UU}(\lambda_Q) & G^s_{UW}(\lambda_Q) & \mathbf{0} & \mathbf{0} \\ G^s_{WU}(\lambda_Q) & G^s_{WW}(\lambda_Q) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\right\} \\
&\quad + \begin{bmatrix} G^{\text{ref}}_{UU}(\lambda_{\text{ref}}) & G^{\text{ref}}_{UW}(\lambda_{\text{ref}}) & \mathbf{0} & \mathbf{0} \\ G^{\text{ref}}_{WU}(\lambda_{\text{ref}}) & G^{\text{ref}}_{WW}(\lambda_{\text{ref}}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
\end{aligned} \tag{290}$$

and $H_{XX}(\mu)$ is the same as for nodal current balance in (277)–(282).

8 Revision History

- **Revision 1** *(October 25, 2018)*
  - Added missing equality constraint for reference voltage angles. See Sections 3.3 and 7.
  - Added missing inequality constraints for bus voltage magnitude limits. See Sections 3.4 and 7.
  - Added missing inequality constraints for branch voltage angle difference limits. See Sections 3.5 and 7.
- **Initial version** *(April 2, 2018)* – Published as "MATPOWER *Technical Note 4*".

Appendix A Scalar Polar Coordinate Derivatives

When using cartesian coordinates for the voltages, the voltage magnitudes and angles are now functions of the cartesian coordinates. Constraints on these functions require their derivatives as well.

Consider a scalar complex voltage v that can be expressed in polar coordinates $|v|$ and θ or cartesian coordinates u and w as:

$$\begin{aligned} v &= |v|e^{j\theta} && (291) \\ &= u + jw && (292) \end{aligned}$$

We also have

$$\begin{aligned} \theta &= \tan^{-1}\frac{w}{u} && (293) \\ |v|^2 &= u^2 + w^2 && (294) \end{aligned}$$

A.1 First Derivatives

Given that

$$\frac{\partial \tan^{-1}(y)}{\partial x} = \frac{1}{1+y^2}\frac{\partial y}{\partial x} \tag{295}$$

we have

$$\frac{\partial \theta}{\partial u} = \frac{1}{1+u^{-2}w^2}\frac{\partial (u^{-1}w)}{\partial u} = \frac{1}{1+u^{-2}w^2}(-u^{-2}w) = -\frac{w}{|v|^2} \tag{296}$$

$$\frac{\partial \theta}{\partial w} = \frac{1}{1+u^{-2}w^2}\frac{\partial (u^{-1}w)}{\partial w} = \frac{1}{1+u^{-2}w^2}u^{-1} = \frac{u}{|v|^2} \tag{297}$$

$$\frac{\partial |v|}{\partial u} = \frac{\partial |v|}{\partial |v|^2}\frac{\partial |v|^2}{\partial u} = \frac{1}{2}(|v|^2)^{-\frac{1}{2}}(2u) = \frac{u}{|v|} \tag{298}$$

$$\frac{\partial |v|}{\partial w} = \frac{\partial |v|}{\partial |v|^2}\frac{\partial |v|^2}{\partial w} = \frac{1}{2}(|v|^2)^{-\frac{1}{2}}(2w) = \frac{w}{|v|} \tag{299}$$

A.2 Second Derivatives

$$\frac{\partial^2\theta}{\partial u^2} = \frac{\partial(-|v|^{-2}w)}{\partial u} = -w(-2|v|^{-3})\frac{u}{|v|} = \frac{2uw}{|v|^4} \tag{300}$$

$$\frac{\partial^2\theta}{\partial w\partial u} = \frac{\partial(|v|^{-2}u)}{\partial u} = \frac{1}{|v|^2} + u\left(\frac{-2}{|v|^3}\right)\frac{u}{|v|} = \frac{|v|^2 - 2u^2}{|v|^4} = \frac{w^2 - u^2}{|v|^4} \tag{301}$$

$$\frac{\partial^2\theta}{\partial u\partial w} = \frac{\partial(-|v|^{-2}w)}{\partial w} = -\frac{1}{|v|^2} - w\left(\frac{-2}{|v|^3}\right)\frac{w}{|v|} = \frac{-|v|^2 + 2w^2}{|v|^4} \tag{302}$$

$$= \frac{w^2 - u^2}{|v|^4} = \frac{\partial^2\theta}{\partial w\partial u} \tag{303}$$

$$\frac{\partial^2\theta}{\partial w^2} = \frac{\partial(|v|^{-2}u)}{\partial w} = u(-2|v|^{-3})\frac{w}{|v|} = -\frac{2uw}{|v|^4} = -\frac{\partial^2\theta}{\partial u^2} \tag{304}$$

$$\frac{\partial^2|v|}{\partial u^2} = \frac{\partial(|v|^{-1}u)}{\partial u} = |v|^{-1} + u(-|v|^{-2})\frac{u}{|v|} = \frac{|v|^2 - u^2}{|v|^3} = \frac{w^2}{|v|^3} \tag{305}$$

$$\frac{\partial^2|v|}{\partial w\partial u} = \frac{\partial(|v|^{-1}w)}{\partial u} = w(-|v|^{-2})\frac{u}{|v|} = -\frac{uw}{|v|^3} \tag{306}$$

$$\frac{\partial^2|v|}{\partial u\partial w} = \frac{\partial(|v|^{-1}u)}{\partial w} = u(-|v|^{-2})\frac{w}{|v|} = -\frac{uw}{|v|^3} = \frac{\partial^2|v|}{\partial w\partial u} \tag{307}$$

$$\frac{\partial^2|v|}{\partial w^2} = \frac{\partial(|v|^{-1}w)}{\partial w} = |v|^{-1} + w(-|v|^{-2})\frac{w}{|v|} = \frac{|v|^2 - w^2}{|v|^3} = \frac{u^2}{|v|^3} \tag{308}$$

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