Power System Structure and Confidentiality Preserving Transformation of Optimal Power Flow Problem

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***Abstract*—In this paper we present a method to transform optimal power flow models to enable the sharing of equivalent data sets while preserving privacy of an original data set. Importantly, the generated models preserve a power system structure with certain characteristics chosen by the developer. The needed transformations are presented on the DC Optimal Power Flow (OPF) model.**

I. Introduction

The electric power grid delivers essential energy to support almost all non-transportation energy needs. It is indisputably important to the functions of society and is considered part of the nation's critical infrastructure. There is considerable concern that the power grid may be vulnerable to and targeted for malicious cyber and physical attacks. These concerns raise conflicting needs in the area of advanced power system analysis and research. As critical energy infrastructure information (CEII), access to true power system data is restricted. At the same time, interest in advancing power system tools with a view toward security has increased. For a researcher working to advance this field, results must be shown on real models to prove credibility.

Unfortunately, these needs and expectations have resulted in the current state in which some researchers with access to CEII data can conduct relevant research using credible models; however, these models and results cannot be independently verified by peers in the field. This goes against traditional scientific principles that call for public verification of results. We argue that there is a fundamental need for new standard publicly-available models that are provably related to credible but secure models. The purpose of this paper is to show that such models can be developed through transformations that yield a new power system model that maintains the privacy of the original data. The transformations map the solution of the new model to the original, establishing a strong and relevant connection between them.

Here we demonstrate the techniques on an optimal power flow (OPF) model as it is the most important problem routinely solved in this industry. The OPF computes the optimal dispatch of resources needed to supply energy while accounting for a range of physical, engineering, and security constraints. It is solved nearly continuously in some form by grid operators to specify and adjust operation of the grid. It is known to be a non-convex problem and much of the current advanced research is directed towards developing efficient methods to find global optima. Here we begin with the analysis of a linearized version of the OPF and show that it is possible to transform a given DC OPF model into a different DC OPF model that relates the optimal solutions through a transformation. We further argue that it is not possible to infer the original model solely from the data of the new transformed model.

In the following sections we present the DC OPF problem specification, the transformations permitting a new DC OPF model, and a specific solved example.

II. DC OPF Problem

The standard DC OPF problem is shown below in (1) and is described in [1], where DC OPF problems were obscured for preserving confidentiality in cloud computing. However, the transformations in [1] do not preserve a typical power system structure, which precludes application of OPF specific solution techniques. The optimization variables in (1) are the powers generated P_g and bus angles δ.

$$\begin{aligned} \min_{P_g,\,\delta}\quad & \tfrac{1}{2} P_g^T \boldsymbol{D_g} P_g + c_g^T P_g \qquad (1) \\ s.t.\quad & -P_g + \boldsymbol{B}\delta = -P_L \\ & \delta_{ref} = 0 \\ & P_{g,min} \le P_g \le P_{g,max} \\ & -P_{flow,max} \le diag(b_{br}) \boldsymbol{A_{inc}} \delta \le P_{flow,max} \end{aligned}$$

The constraints in (1) are power balance at each bus, the reference bus angle equal to 0°, power generation upper and lower limits, and power flow limits in both directions on transmission lines. Primarily, the analysis in this paper will be on DC OPF problems with linear cost functions. Therefore the quadratic cost terms in (1) will be removed, and the problem will be rewritten in general form. Notation is adopted from prior literature [2,3] that examines linear transformation problems.

$$\begin{aligned} &\min_{x} \quad c^T x \\ s.t. \quad &\boldsymbol{M} x = b \\ &\boldsymbol{I_{sl}} x \geq 0 \end{aligned} \tag{2}$$

In (2), P_g and δ are free variables. Slack variables x_{sl} are included to convert inequality constraints to equality constraints. They are nonnegative, $x_{sl} \geq 0$, and are enforced by $\boldsymbol{I_{sl}} x \geq 0$. The optimization variables are $x = \begin{bmatrix} P_g^T & \delta^T & x_{sl}^T \end{bmatrix}^T$. The constraint matrix $\boldsymbol{M}$ and vector b have typical topologies as seen in Figure 1 which displays the IEEE 14-bus test system [5].

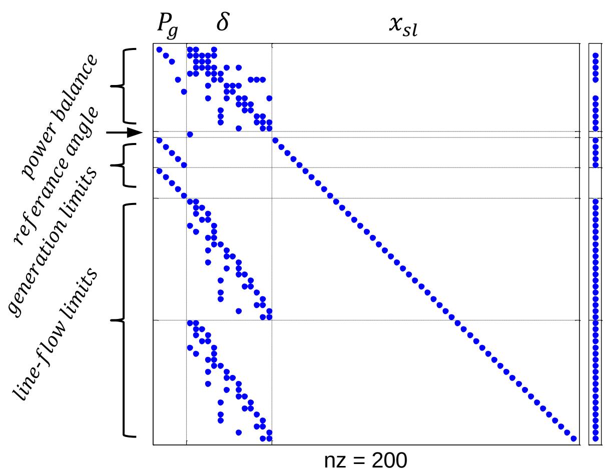

Figure 1: $\boldsymbol{M}$ matrix and $\boldsymbol{b}$ vector for IEEE 14-bus system

Matrix $\boldsymbol{M}$ has $m = n_{bus} + 2n_{gen} + 2n_{line} + 1$ rows and $n = n_{bus} + 3n_{gen} + 2n_{line}$ columns. Therefore there are $n - m = n_{gen} - 1$ more columns than rows in $\boldsymbol{M}$. It should be emphasized that the number of rows m is less than number of columns n, as it is critical for the transformation analysis. Note that the $\boldsymbol{M}$ matrix has full-rank in typical DC OPF problems and is therefore assumed to be full-rank.

III. TRANSFORMATION PROBLEM

In this section we begin the transformation procedure starting from one DC OPF problem to a completely new problem. To begin with, two different DC OPF problems are constructed, with the first in (2) and the second in (3).

$$\begin{aligned} &\min_{x'} \quad c'^T x' \\ s.t. \quad &\boldsymbol{M}' x' = b' \\ &\boldsymbol{I_{sl}} x' \geq 0 \end{aligned} \tag{3}$$

For simplicity, it is assumed the two DC OPF problems possess the same number of generators, buses and lines. The two models differ by having different bus and line topologies as well as having numerically different component values such as line susceptances, generator cost coefficients, etc. The optimal solutions x^* and x'^* to DC OPF problems (2) and (3) respectively, must be known before proceeding with the transformation.

We next show a bidirectional linear transformation between (2) and (3). The goal is to construct the transformation matrices $\boldsymbol{P}$ and $\boldsymbol{T}$ and vectors w and r such that the following relationships hold.

$$\boldsymbol{M}' = \boldsymbol{PMT} \tag{4a}$$

$$c' = \boldsymbol{T}^T c + \boldsymbol{M}'^T w \tag{4b}$$

$$b' = \boldsymbol{P} b + \boldsymbol{M}' r \tag{4c}$$

$$x^* = \boldsymbol{T}(x'^* - r) \tag{4d}$$

The above relationships all have specific purposes. Relationship (4a) enforces a linear transformation between the constraint matrix $\boldsymbol{M}$ in (2) and the constraint matrix $\boldsymbol{M}'$ in (3). This is done by multiplying $\boldsymbol{M}$ on the left and right by some transformation matrices $\boldsymbol{P}$ and $\boldsymbol{T}$.

Relationship (4b) enforces a linear transformation between the cost coefficient vector c in (2) and the cost coefficient vector c' in (3). A linear combination of the rows in constraint matrix $\boldsymbol{M}'$ is added to $\boldsymbol{T}^T c$ in order to satisfy (4b). This is justified because adding $\boldsymbol{M}'^T w$ to the cost function does nothing more than add a constant $b'^T w$, $c'^T x' = (c^T \boldsymbol{T} + w^T \boldsymbol{M}') x' = c^T \boldsymbol{T} x' + w^T b'$.

Relationship (4c) enforces a linear transformation between the constraint vector b in (2) and the constraint vector b' in (3).

Lastly, relationship (4d) enforces a linear transformation between the optimal solution x^* in (2) and the optimal solution x'^* in (3). In [1] and [2], the matrix $\boldsymbol{T}$ was required to be a positive monomial matrix, (i.e. a matrix containing exactly one positive entry per row and column, with the remainder of the entries being zero), and vector r was required to have positive entries. By defining $\boldsymbol{T}$ and r this way, it can be shown that if $\boldsymbol{I_{sl}}\, x \geq 0$ then it is guaranteed that $\boldsymbol{I_{sl}}\, x' \geq 0$ as well. The requirements on $\boldsymbol{T}$ and r from [1] and [2] will be removed now, as they restrict the ability to transform to a new system with an OPF structure.

The transformation between the DC OPF problems (2) and (3) is developed by determining appropriate matrices $\boldsymbol{P}$ and $\boldsymbol{T}$ and vectors w and r such that the relationships (4a)−(4d) hold. We next exploit the relationships in (4a)−(4d) and the degrees of freedom inherent in the transformation (i.e. the null-spaces of the DC OPF problems' matrices) to describe an appropriate choice for these matrices and vectors. The specifics of each transformation are detailed next.

To begin, the relationship (4a) can be enforced by defining $\boldsymbol{T}$ as the sum of two terms, $\boldsymbol{T} = \boldsymbol{T_0} + \boldsymbol{NQ}$, where the $n \times (n-m)$ matrix $\boldsymbol{N}$ is the null-space of $\boldsymbol{M}$, i.e. $\boldsymbol{MN} = \boldsymbol{0}_{m \times (n-m)}$. Given any nonsingular $\boldsymbol{P}$, one can solve

$\boldsymbol{T_0} = \boldsymbol{M}^{\dagger}\boldsymbol{P}^{-1}\boldsymbol{M'}$, where † denotes the Moore-Penrose pseudoinverse [4]. This solution structure on $\boldsymbol{T}$ satisfies (4a). It is necessary for $\boldsymbol{T}$ to be full-rank, therefore matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ must be full-rank.

Next, relationship (4b) is expanded by substituting $\boldsymbol{T} = \boldsymbol{T_0} + \boldsymbol{NQ}$ as shown in (5), where matrix $\boldsymbol{Q}$ and vector w are unknown.

$$c' = \boldsymbol{T_0}^T c + \boldsymbol{Q}^T\boldsymbol{N}^T c + \boldsymbol{M'}^T w \tag{5}$$

The procedure goes as follows, w is solved as a function of $\boldsymbol{Q}$ in (6) by taking the pseudoinverse of $\boldsymbol{M'}^T$.

$$w = \boldsymbol{M'}^{T\dagger}(c' - \boldsymbol{T_0}^T c - \boldsymbol{Q}^T\boldsymbol{N}^T c) \tag{6}$$

Next w is substituted back into (5) which can be rewritten as in (7).

$$\left(I - \boldsymbol{M'}^T\boldsymbol{M'}^{T\dagger}\right)(c' - \boldsymbol{T_0}^T c) = \left(I - \boldsymbol{M'}^T\boldsymbol{M'}^{T\dagger}\right)\boldsymbol{Q}^T\boldsymbol{N}^T c \tag{7}$$

Note that $\boldsymbol{M'}^T\boldsymbol{M'}^{T\dagger} \neq I_{n\times n}$ and $\boldsymbol{M'}^{T\dagger}\boldsymbol{M'}^T = I_{m\times m}$, therefore it follows that $\left(I - \boldsymbol{M'}^T\boldsymbol{M'}^{T\dagger}\right)\boldsymbol{M'}^T = \boldsymbol{0}_{n\times m}$. The $n\times n$ matrix $I - \boldsymbol{M'}^T\boldsymbol{M'}^{T\dagger}$ has rank of order $n-m$, and can be rewritten as $I - \boldsymbol{M'}^T\boldsymbol{M'}^{T\dagger} = \boldsymbol{N'}\boldsymbol{N'}^T$, where the $n\times(n-m)$ matrix $\boldsymbol{N'}$ is the null-space of $\boldsymbol{M'}$, i.e. $\boldsymbol{M'}\boldsymbol{N'} = \boldsymbol{0}_{m\times(n-m)}$. Note that $\boldsymbol{N'}^{\dagger} = \boldsymbol{N'}^T$, $\boldsymbol{N'}\boldsymbol{N'}^T \neq I_{n\times n}$ and $\boldsymbol{N'}^T\boldsymbol{N'} = I_{(n-m)\times(n-m)}$.

Before solving for $\boldsymbol{Q}$ by using (7) and the above observations, it should be emphasized that $\boldsymbol{T} = \boldsymbol{T_0} + \boldsymbol{NQ}$ must have full-rank of n. $\boldsymbol{T_0}$ will have rank m as long as $\boldsymbol{P}$ is full-rank m; therefore, $\boldsymbol{NQ}$ must have rank $n-m$. It becomes necessary to split $\boldsymbol{Q}$ into the sum of two $(n-m)\times n$ terms $\boldsymbol{Q} = \boldsymbol{Q_1} + \boldsymbol{Q_2}$. The first of these two terms $\boldsymbol{Q_1}$ will be solved by using (7) and the prior observations.

$$\boldsymbol{Q_1} = (c^T\boldsymbol{N})^{\dagger} c'^T\boldsymbol{N'}\boldsymbol{N'}^T \tag{8}$$

Setting $\boldsymbol{Q} = \boldsymbol{Q_1}$ now satisfies (7). With w solved as a function of $\boldsymbol{Q}$ using (6), relationship (4b) will be satisfied. As it turns out however, $\boldsymbol{TQ_1}$ is rank 1, and therefore $\boldsymbol{T}$ is still lacking $n-m-1$ in its rank. Fortunately the remaining $n-m-1$ rank can be acquired by the second term $\boldsymbol{Q_2}$. Observe in (5) that $\boldsymbol{Q}^T$ multiplies an $(n-m)\times 1$ vector $\boldsymbol{N}^T c$. Define $\tilde{\boldsymbol{N}}$ as the $(n-m)\times(n-m-1)$ null-space of $c^T\boldsymbol{N}$, such that $c^T\boldsymbol{N}\tilde{\boldsymbol{N}} = 0_{1\times(n-m-1)}$. Define an $(n-m-1)\times n$ matrix $\boldsymbol{V}$ which can be any full-rank matrix. Finally define $\boldsymbol{Q_2} = \tilde{\boldsymbol{N}}\boldsymbol{V}$, which yields $\boldsymbol{T} = \boldsymbol{T_0} + \boldsymbol{N}(\boldsymbol{Q_1} + \boldsymbol{Q_2})$ to be a full-rank matrix.

The two vector relationships (4c) and (4d) have yet to be satisfied, but can be satisfied by appropriately determining the $n\times 1$ vector r. Define r as the sum of two terms, $r = r_0 + \boldsymbol{N'}q$. Rearrange (4c) to solve for r_0 as shown in (9).

$$r_0 = \boldsymbol{M'}^{\dagger}(b' - \boldsymbol{P}b) \tag{9}$$

Setting $r = r_0$ will now satisfy (4c), but not (4d). The last remaining variable to be solved for is q, which can be obtained by substituting $r = r_0 + \boldsymbol{N'}q$ and rearranging (4d).

$$q = \boldsymbol{N'}^{\dagger}(x'^* - \boldsymbol{T}^{-1}x^* - r_0) \tag{10}$$

Substitute q from (10) into $r = r_0 + \boldsymbol{N'}q$. This solution for r satisfies both (4c) and (4d). At this point, all four relationships (4a)−(4d) have been satisfied by appropriately determining the transformation matrix $\boldsymbol{T}$ and vectors w and r. The transformation solution is not unique, as there is some flexibility in choosing any full-rank matrix $\boldsymbol{P}$ and also any full-rank matrix $\boldsymbol{V}$ as described above.

IV. Example

We next present a numeric example of the transformation method described in Section III. Specifically, we transform between two OPF problems derived from the IEEE 14-bus system [5]. The first OPF problem has the same network topology and line susceptances (calculated as the reciprocal of the line reactance and neglecting line resistance) as the standard IEEE 14-bus system and has a 100 MVA base power. The one-line diagram for this system is shown in Figure 2. To enable two-dimensional plotting of the feasible space of generator power injections (detailed in the Appendix), this example considers an OPF problem with only three generators, as opposed to five generators in the standard IEEE 14-bus system. The coefficients for the linear generator cost functions are given in Table 1. All generators have lower generation limits of zero and upper generation limits specified in Table 1. Line-flow limits of 100 MW are enforced on all lines. Load demands are the same as those specified for the standard IEEE 14-bus system.

Generator	Cost Coefficient [$\$/MWh$]	Upper Generation Limit [MW]
1	20	330
2	30	140
8	25	50

Table 1: Generator Data for First OPF Problem

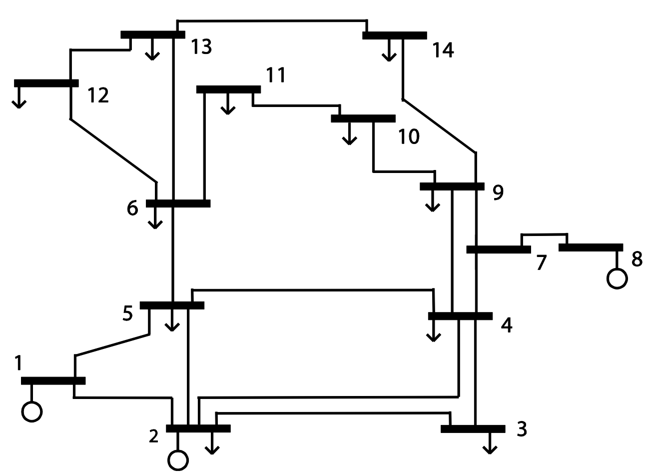

Figure 2: One-Line Diagram for First OPF Problem

The second OPF problem has the same number of buses, lines and generators as the first OPF problem, but has different network topology, line susceptances, generator and line-flow limits, load demands and generator costs. The coefficients for the linear generator cost functions are given in Table 2, load demands are given in Table 3 and line susceptances are given in Table 4. All line-flows are limited to 90 MW. The network topology of the second OPF problem is shown in the one-line diagram in Figure 3.

Generator	Cost Coefficient [$\$/MWh$]	Upper Generation Limit [MW]
1	25	150
8	10	40
13	20	150

Table 2: Generator Data for Second OPF Problem

Bus	1	2	3	4	5	6	7
Demand [MW]	0	20.2	87.8	49.9	7.9	10.9	0

Bus	8	9	10	11	12	13	14
Demand [MW]	0	28.7	9.5	3.5	5.6	14.3	15.4

Table 3: Load Demands for Second OPF Problem

From Bus	1	1	8	2	2	8	4
To Bus	12	5	3	4	5	4	5
Suscept-ance [$p.u.$]	17.04	3.85	5.76	5.79	5.49	5.63	22.27

From Bus	4	4	5	6	6	6	7
To Bus	7	10	11	11	12	13	8
Suscept-ance [$p.u.$]	5.27	1.85	3.54	4.29	3.96	8.32	5.82

From Bus	7	9	9	10	12	6
To Bus	14	11	14	11	13	14
Suscept-ance [$p.u.$]	9.85	13.0	4.32	6.01	5.02	3.44

Table 4: Line Susceptances for Second OPF Problem

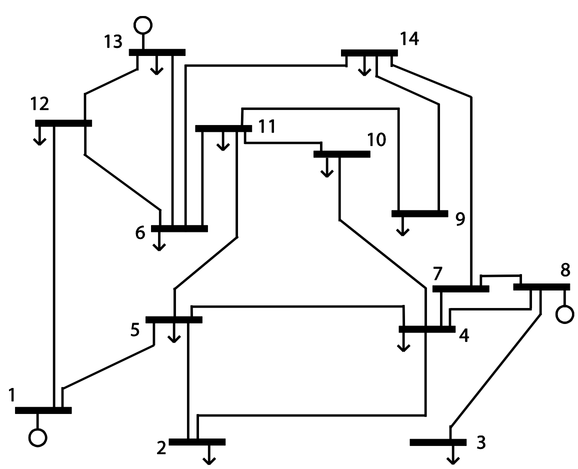

Figure 3: One-Line Diagram for Second OPF Problem

The feasible spaces for generator power injections in the first and second OPF problems are shown in Figures 4 and 5, respectively. The details for reducing these OPF problems to two optimization variables (generator power injections P_{g1} and P_{g2}) and plotting the optimization feasible space are outlined in the Appendix. The contour lines illustrate the linear cost function, with color blue representing lower cost. Using the procedure detailed in Section III, a transformation (i.e. a set of matrices $\boldsymbol{P}$ and $\boldsymbol{T}$ and vectors w and r) is generated that satisfies relationships (4a)−(4d).

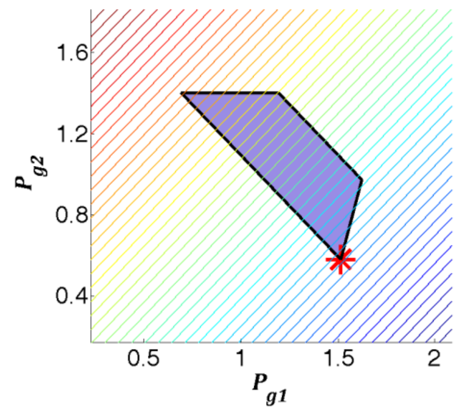

Figure 4: Feasible Space of Power Generation for First OPF Problem

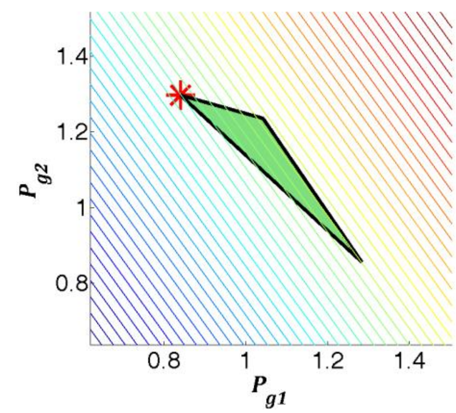

Figure 5: Feasible Space of Power Generation for Second OPF Problem

The feasible space in Figure 4 corresponds to the first OPF problem having the formulation in (2). The feasible space in Figure 5 corresponds to the second OPF problem having the formulation in (3). The procedure used for plotting the feasible spaces is described in the Appendix.

In Figure 6, the transformation is applied to the first OPF problem (2) and is plotted in the coordinate system consistent with the second OPF problem (3). That is to say, the larger purple colored polytope in Figure 6 corresponds to the feasible space of the problem formulation shown below in (11b). The formulation in (11a) displays the transformation taking place on (2) to convert to (3). The formulations in (11a) and (11b) are equivalent, and the optimal solution x^* of (2) can be recovered from either (11a) or (11b).

Notice the formulation in (11b) is nearly identical to the formulation in (3), excluding the inequality constraints $\boldsymbol{I_{sl}}\boldsymbol{T}x' \geq \boldsymbol{I_{sl}}\boldsymbol{T}r$. These inequality constraints are replaced by $\boldsymbol{I_{sl}}x' \geq 0$ to completely transform (2) to (3), which does not change the optimal solution x'^{*}. The feasible space of (3) is shown as the green colored polytope in Figure 6.

$$\min_{x} \quad (c^T\boldsymbol{T} + w^T\boldsymbol{PMT})(\boldsymbol{T}^{-1}x + r) \tag{11a}$$
$$s.t. \quad \boldsymbol{PMT}(\boldsymbol{T}^{-1}x + r) = \boldsymbol{P}b + \boldsymbol{PMT}r$$
$$\boldsymbol{I_{sl}}x \geq 0$$
$$\Updownarrow$$
$$\min_{x'} \quad c'x' \tag{11b}$$
$$s.t. \quad \boldsymbol{M}'x' = \boldsymbol{b}'$$
$$\boldsymbol{I_{sl}}\big(\boldsymbol{T}(x' - r)\big) \geq 0$$

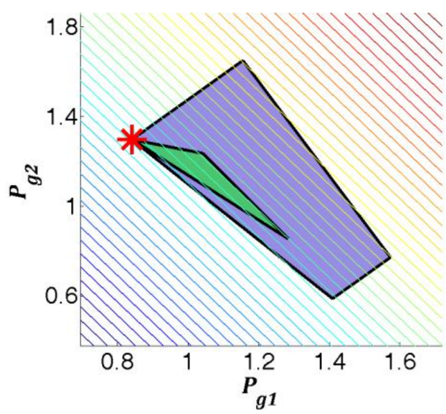

Figure 6: Feasible Space of Both OPF Problems in Coordinates of Second OPF Problem

Both feasible spaces in Figure 6 return the same optimal solution, indicated by the red star in the figure. It is therefore true that the optimal solution x'^{*} to (3) can be used to recover the optimal solution to (2) by $x^{*} = \boldsymbol{T}(x'^{*} - r)$. The small-scale OPF in this example was chosen so that plotting the feasible space in two-dimensions was possible; however, the transformation procedure outlined in Section III is applicable to any larger DC OPF having linear cost function. This example has demonstrated the method for transforming one DC OPF problem to another while preserving the optimal solution. In the next section, DC OPF problems with quadratic and piecewise linear cost functions are analyzed.

V. Quadratic and Piecewise-Linear Cost Functions

The DC OPF model used to develop the transformation in Section III requires linear cost functions on active power generation. Exploratory work has investigated extension of this transformation to quadratic and piecewise-linear cost functions.

A. *Quadratic Cost Function*

Section III detailed the method for creating a transformation between two DC OPF problems with linear cost functions. In this section, we investigate what is necessary in order to perform a transformation between two DC OPF problems having quadratic cost functions. The requirements for the transformation are more strict than the linear cost function case, and it will be shown only systems meeting a certain required property can be transformed to one another. If two systems do not meet this requirement, they can only be transformed to one another by introducing an additional variable/degree of freedom that was not needed in the case of the linear cost function.

Two DC OPF problems with quadratic cost functions are shown in (12) and (13).

$$\min_{x} \ \tfrac{1}{2}x^T\boldsymbol{D}x + c^Tx \tag{12}$$
$$s.t. \quad \boldsymbol{M}x = b$$
$$\boldsymbol{I_{sl}}x \geq 0$$

$$\min_{x'} \ \tfrac{1}{2}x'^T\boldsymbol{D}'x' + c'^Tx' \tag{13}$$
$$s.t. \quad \boldsymbol{M}'x' = b'$$
$$\boldsymbol{I_{sl}}x' \geq 0$$

The goal is to determine a set of transformation matrices and vectors that establish a bidirectional transformation between (12) and (13). Relationships (14a)- (14e) must hold in order for the transformation to be complete. Transformation matrices $\boldsymbol{P}$ and $\boldsymbol{T}$ and vectors r and w are analogous to those defined in Section III. The matrix $\boldsymbol{W}$ in (14b) and (14c) results from the additional degrees of freedom in OPF problems with quadratic cost functions.

$$\boldsymbol{M}' = \boldsymbol{PMT} \tag{14a}$$
$$\boldsymbol{D}' = \boldsymbol{T}^T\boldsymbol{DT} + 2\boldsymbol{M}'^T\boldsymbol{WM}' \tag{14b}$$
$$c' = \boldsymbol{T}^Tc - \boldsymbol{T}^T\boldsymbol{DT}r - 2\boldsymbol{M}'^T\boldsymbol{W}b' + \boldsymbol{M}'^Tw \tag{14c}$$
$$b' = \boldsymbol{P}b + \boldsymbol{M}'r \tag{14d}$$
$$x^{*} = \boldsymbol{T}(x'^{*} - r) \tag{14e}$$

Relationships (14a), (14d) and (14e) are exactly the same as relationships (4a), (4c) and (4d) in the linear cost function case. There is now, however, an additional relationship, (14b), which enforces a transformation between the quadratic cost matrix $\boldsymbol{D}$ in (12) and the quadratic cost matrix $\boldsymbol{D}'$ in (13). In (14b), a quadratic combination of the rows in constraint matrix $\boldsymbol{M}'$ is added to $\boldsymbol{T}^T\boldsymbol{DT}$. Adding $2\boldsymbol{M}'^T\boldsymbol{WM}'$ to the quadratic cost matrix in (14b) and subtracting $2\boldsymbol{M}'^T\boldsymbol{W}b'$ from the linear cost vector in (14c), effectively just adds a constant to the cost function.

$$0 = (x'^T\boldsymbol{M}'^T - b'^T)\boldsymbol{W}(\boldsymbol{M}'x' - b') \tag{15}$$
$$= x'^T\boldsymbol{M}'^T\boldsymbol{WM}'x' - 2b'^T\boldsymbol{WM}'x' + b'^T\boldsymbol{W}b'$$

The cost function in (13) effectively has a constant $x'^T \boldsymbol{M'}^T \boldsymbol{W} \boldsymbol{M'} x' - 2b'^T \boldsymbol{W} \boldsymbol{M'} x' = -b'^T \boldsymbol{W} b'$ added to it, as evident in (15).

The relationship between the linear cost vector c in (12) and linear cost vector c' in (13), is shown in (14c). Similar to relationship (4b), a linear combination of the rows in constraint matrix $\boldsymbol{M'}$ is added as $\boldsymbol{M'}^T w$, which effectively adds a constant to the cost function. There is one additional term in (14c) that has yet to be explained, that being $-\boldsymbol{T}^T \boldsymbol{D} \boldsymbol{T} r$. Consider substituting $x = \boldsymbol{T}(x' - r)$ into the quadratic cost term $\frac{1}{2} x^T \boldsymbol{D} x$ in (12).

$$\frac{1}{2} x^T \boldsymbol{D} x = \frac{1}{2}(x'^T - r^T)\boldsymbol{T}^T \boldsymbol{D} \boldsymbol{T}(x' - r) \tag{16}$$
$$= \frac{1}{2} x'^T \boldsymbol{T}^T \boldsymbol{D} \boldsymbol{T} x' - r^T \boldsymbol{T}^T \boldsymbol{D} \boldsymbol{T}\, x' + \frac{1}{2} r^T \boldsymbol{T}^T \boldsymbol{D} \boldsymbol{T} r$$

As shown in (16), quadratic, linear and constant terms are created after making the substitution. Therefore $-\boldsymbol{T}^T \boldsymbol{D} \boldsymbol{T} r$ in (14c) originates from the cross coupling terms of the quadratic cost function transformation. The constant term $\frac{1}{2} r^T \boldsymbol{T}^T \boldsymbol{D} \boldsymbol{T} r$ is dropped altogether, as it has no impact on results of the optimization in (13).

The transformation procedure between two DC OPF problems having quadratic cost functions is detailed next. The procedure is started the same as the linear cost function case, where relationship (14a) is enforced by defining $\boldsymbol{T}$ as the sum of two terms, $\boldsymbol{T} = \boldsymbol{T_0} + \boldsymbol{N} \boldsymbol{Q}$, where $\boldsymbol{T_0} = \boldsymbol{M}^\dagger \boldsymbol{P}^{-1} \boldsymbol{M'}$ and the $n \times (n - m)$ matrix $\boldsymbol{N}$ is the null-space of $\boldsymbol{M}$, i.e. $\boldsymbol{M} \boldsymbol{N} = \boldsymbol{0}_{m \times (n-m)}$. With relationship (14a) satisfied, we move on to relationship (14b).

The procedure goes as follows, (14b) is rearranged by taking the pseudoinverse of $\boldsymbol{M'}^T$ and $\boldsymbol{M'}$ to solve for $\boldsymbol{W}$ as a function of $\boldsymbol{T}$ as shown in (17).

$$\boldsymbol{W} = \frac{1}{2} \boldsymbol{M'}^{T^\dagger} (\boldsymbol{D'} - \boldsymbol{T}^T \boldsymbol{D} \boldsymbol{T}) \boldsymbol{M'}^\dagger \tag{17}$$

Next, $\boldsymbol{W}$ from (17) and $\boldsymbol{T} = \boldsymbol{M}^\dagger \boldsymbol{P}^{-1} \boldsymbol{M'} + \boldsymbol{N} \boldsymbol{Q}$ are substituted back into (14b), which can be rewritten as in (18). Recall that $\boldsymbol{M'}^T \boldsymbol{M'}^{T^\dagger} \neq I_{n \times n}$ and $\boldsymbol{M'}^{T^\dagger} \boldsymbol{M'}^T = I_{m \times m}$.

$$\boldsymbol{D'} - \boldsymbol{Q}^T \boldsymbol{N}^T \boldsymbol{D} \boldsymbol{N} \boldsymbol{Q} - \boldsymbol{M'}^T \boldsymbol{M'}^{T^\dagger} (\boldsymbol{D'} - \boldsymbol{Q}^T \boldsymbol{N}^T \boldsymbol{D} \boldsymbol{N} \boldsymbol{Q}) \boldsymbol{M'}^\dagger \boldsymbol{M'}$$
$$= \boldsymbol{M'}^T \boldsymbol{P}^{T^{-1}} \boldsymbol{M}^{T^\dagger} \boldsymbol{D} \boldsymbol{N} \boldsymbol{Q} (I - \boldsymbol{M'}^\dagger \boldsymbol{M'}) \tag{18}$$
$$+ \left(I - \boldsymbol{M'}^T \boldsymbol{M'}^{T^\dagger}\right) \boldsymbol{Q}^T \boldsymbol{N}^T \boldsymbol{D} \boldsymbol{M}^\dagger \boldsymbol{P}^{-1} \boldsymbol{M'}$$

Note that $I - \boldsymbol{M'}^\dagger \boldsymbol{M'} = I - \boldsymbol{M'}^T \boldsymbol{M'}^{T^\dagger} = \boldsymbol{N'} \boldsymbol{N'}^T$, where the $n \times (n - m)$ matrix $\boldsymbol{N'}$ is the null-space of $\boldsymbol{M'}$, i.e. $\boldsymbol{M'} \boldsymbol{N'} = \boldsymbol{0}_{m \times (n-m)}$. Recall that $\boldsymbol{N'}^\dagger = \boldsymbol{N'}^T$, $\boldsymbol{N'} \boldsymbol{N'}^T \neq I_{n \times n}$ and $\boldsymbol{N'}^T \boldsymbol{N'} = I_{(n-m) \times (n-m)}$.

Matrices $\boldsymbol{Q}$ and $\boldsymbol{P}^{-1}$ are unknown in (18). However, (18) has a special structure that allows solving for $\boldsymbol{P}^{-1}$ as a function of $\boldsymbol{Q}$. Define $\widetilde{\boldsymbol{N}}$ as the $m \times (2m - n)$ null-space of $\boldsymbol{N}^T \boldsymbol{D} \boldsymbol{M}^\dagger$, such that $\boldsymbol{N}^T \boldsymbol{D} \boldsymbol{M}^\dagger \widetilde{\boldsymbol{N}} = \boldsymbol{0}_{(n-m) \times (2m-n)}$. Define a $(2m - n) \times m$ matrix $\boldsymbol{V}$ which can be any full-rank matrix. A full-rank solution for $\boldsymbol{P}^{-1}$ as a function of $\boldsymbol{Q}$ is deduced from (18) and is shown in (19).

$$\boldsymbol{P}^{-1} = \widetilde{\boldsymbol{N}} \boldsymbol{V} + \tag{19}$$
$$(\boldsymbol{N} \boldsymbol{D} \boldsymbol{M}^\dagger)^\dagger \boldsymbol{Q}^{T^\dagger} \boldsymbol{N'} \boldsymbol{N'}^T (\boldsymbol{D'} - \boldsymbol{Q}^T \boldsymbol{N}^T \boldsymbol{D} \boldsymbol{N} \boldsymbol{Q}) \boldsymbol{M'}^\dagger$$

Substituting the solution for $\boldsymbol{P}^{-1}$ in (19) into (18), reduces (18) to (20).

$$\boldsymbol{N'} \boldsymbol{N'}^T (\boldsymbol{D'} - \boldsymbol{Q}^T \boldsymbol{N}^T \boldsymbol{D} \boldsymbol{N} \boldsymbol{Q}) \boldsymbol{N'} \boldsymbol{N'}^T = \boldsymbol{0}_{n \times n} \tag{20}$$

In (20), the only unknown variable is matrix $\boldsymbol{Q}$. The solution for $\boldsymbol{Q}$ in (20) is not unique, but is rather adjustable by any $(n - m) \times (n - m)$ orthogonal matrix $\boldsymbol{R}$, as shown in (21).

$$\boldsymbol{Q} = (\boldsymbol{N}^T \boldsymbol{D} \boldsymbol{N})^{-1/2} \boldsymbol{R} \, (\boldsymbol{N'}^T \boldsymbol{D'} \boldsymbol{N'})^{1/2} \boldsymbol{N'}^T \tag{21}$$

In (21), the exponent $1/2$ denotes a matrix square root, such that $\boldsymbol{Y} = \boldsymbol{X}^{1/2}$ and $\boldsymbol{Y}\boldsymbol{Y} = \boldsymbol{X}$, and the exponent $-1/2$ denotes the inverse of the matrix square root. With $\boldsymbol{Q}$ from (21) and $\boldsymbol{T} = \boldsymbol{M}^\dagger \boldsymbol{P}^{-1} \boldsymbol{M'} + \boldsymbol{N} \boldsymbol{Q}$, matrix $\boldsymbol{W}$ can be solved using (17), and relationship (14b) is satisfied.

Relationships (14d) and (14e) are exactly the same as (4c) and (4d) in the case of a linear cost function. As described in Section III, the relationships are satisfied by defining r as the sum of two terms, $r = r_0 + \boldsymbol{N'} q$. It is necessary for $r_0 = \boldsymbol{M'}^\dagger (b' - \boldsymbol{P} b)$ and $q = \boldsymbol{N'}^\dagger (x'^* - \boldsymbol{T}^{-1} x^* - r_0)$ in order to satisfy (14d) and (14e). With proper cancelation, r can be rewritten in (22).

$$r = \boldsymbol{M'}^\dagger b' - \boldsymbol{T}^{-1} x^* + \boldsymbol{N'} \boldsymbol{N'}^T x'^* \tag{22}$$

The last remaining relationship to be satisfied is (14c), which equates the linear cost coefficients of DC OPF problems (12) and (13). Relationship (14c) will next be examined to determine what property is required for two systems, with quadratic cost functions, to be able to transform to one another. Rearrange (14c) to solve for w as a function of $\boldsymbol{Q}$ in (23) by taking the pseudoinverse of $\boldsymbol{M'}^T$.

$$w = \boldsymbol{M'}^{T^\dagger} (c' - \boldsymbol{T}^T c + \boldsymbol{T}^T \boldsymbol{D} \boldsymbol{T} + 2 \boldsymbol{M'}^T \boldsymbol{W} b) \tag{23}$$

By substituting r from (22), substituting $\boldsymbol{T}^T \boldsymbol{D} \boldsymbol{T} = \boldsymbol{D'} - 2 \boldsymbol{M'}^T \boldsymbol{W} \boldsymbol{M'}$ and substituting $\boldsymbol{T} = \boldsymbol{M}^\dagger \boldsymbol{P}^{-1} \boldsymbol{M'} + \boldsymbol{N} \boldsymbol{Q}$, (14c) can be reduced to reveal a necessary property in order for two systems, having quadratic cost functions, to be able to transform to one another. This necessary property is shown in (24).

$$(\boldsymbol{N}^T \boldsymbol{D} \boldsymbol{N})^{-1/2} \boldsymbol{N}^T (c + \boldsymbol{D} x^*) \tag{24}$$
$$= \boldsymbol{R} \, (\boldsymbol{N'}^T \boldsymbol{D'} \boldsymbol{N'})^{-1/2} \boldsymbol{N'}^T (c' + \boldsymbol{D'} x'^*)$$

The two DC OPF problems in (12) and (13) must satisfy the given vector relationship (24) in order for (14c) to be satisfied. The only degree of freedom in (24) is the $(n-m)\times(n-m)$ orthogonal matrix $\boldsymbol{R}$. All other variables are predetermined by the DC OPF models in (12) and (13). It is acceptable to use a complex valued orthogonal matrix $\boldsymbol{R}$ if needed. If no $\boldsymbol{R}$ matrix can be found, then an additional degree of freedom must be introduced. Quite simply, the last degree of freedom needed would be an $n\times 1$ vector $\tilde{c}$, such that (25) is satisfied.

$$c' = \boldsymbol{T}^T c - \boldsymbol{T}^T \boldsymbol{D}\boldsymbol{T} r - 2\boldsymbol{M'}^T \boldsymbol{W} b' + \tilde{c} \tag{25}$$

At this point, the four relationships (14a), (14b), (14d) and (14e) have been satisfied by appropriately determining the transformation matrices $\boldsymbol{T}$ and $\boldsymbol{W}$ and vector r. If relationship (14c) cannot be solved by determining the orthogonal matrix $\boldsymbol{R}$, then an additional degree of freedom $\tilde{c}$ is introduced such that (25) is satisfied instead of (14c). With this approach, two DC OPF models with quadratic cost functions can be transformed to one another.

B. Piecewise-Linear Cost Function

Convex piecewise-linear cost functions are often used in DC OPF problems, particularly in electricity market contexts. Available literature on the piecewise-linear formulation can be reviewed in [6]. Consider a piecewise-linear cost function for generator i with r_i linear segments specified by slopes $m_{i,1},..., m_{i,r_i}$ and breakpoints (a_{ij}, b_{ij}), $j = 1,..., r_i$, where a_{ij} is the power generation coordinate and b_{ij} is the cost coordinate for the j^{th} breakpoint of generator i. With these specifications, the cost of power generation at the i^{th} generator becomes $C_g(P_{g,i})$ as shown in (26).

$$C_g(P_{g,i}) = \begin{cases} m_{i,1}(P_{g,i} - a_{i,1}) + b_{i,1}, & P_{g,i} \le a_{i,1} \\ m_{i,2}(P_{g,i} - a_{i,2}) + b_{i,2}, & a_{i,1} < P_{g,i} \le a_{i,2} \\ \vdots & \vdots \\ m_{i,r_i}(P_{g,i} - a_{i,r_i}) + b_{i,r_i}, & a_{i,r_i} \le P_{g,i} \end{cases} \tag{26}$$

Convex piecewise-linear cost functions can be implemented as a linear program using a set of linear inequality constraints. Specifically, define a scalar variable β_i for each generator. Then the piecewise-linear cost curves are implemented using the linear program in (27), which is the piecewise-linear modification of the formulation in (1).

Incorporation of this formulation for convex piecewise-linear cost functions does not change the fundamental characteristics of the DC OPF problem since inequality constraints are already allowed in the DC OPF formulation (1). The formulation for piecewise-linear cost functions has a linear objective, and therefore the transformation method described in Section III can be directly applied to this modified problem.

$$\min_{P_g,\,\delta,\,\beta} \sum_{i=1}^{n_{gen}} \beta_i \tag{27}$$

$$\begin{aligned} s.t. \quad & -P_g + \boldsymbol{B}\delta = -P_L \\ & \delta_{ref} = 0 \\ & P_{g,min} \le P_g \le P_{g,max} \\ & -P_{flow,max} \le diag(b_{br})\boldsymbol{A_{inc}}\delta \le P_{flow,max} \\ & \{\beta_i \ge m_{g,t}(P_{g,i} - a_{i,t}) + b_{i,t} \quad \forall\, t = 1,...,r_i\} \quad \forall\, i = 1,...,n_{gen} \end{aligned}$$

VI. Conclusion

This paper has outlined a transformation method between two DC optimal power flow (OPF) problems, and by extension to a family of problems, which preserves a mapping between optimal solutions. The transformation method was first developed for DC OPF problems having linear cost functions, and the method was demonstrated on an example using a modified version of the IEEE 14-bus system. Next, the transformation method was developed for DC OPF problems having quadratic cost functions, and lastly was developed for DC OPF problems having piecewise-linear cost functions. The needed next steps include the extension to AC OPF models, and a means to compare the computational complexity of the original and transformed models for the nonlinear AC OPF models.

Prior related work has examined transforming/masking OPF problems for purposes of preserving system confidentiality in cloud computing [1]. However, in that work the transformed/masked problem did not resemble a typical power system structure. For the purpose of cloud computing, we do not require the model to have a power system structure. The methods detailed in this paper prove the existence of transformations that preserve power system structure by constructively calculating the transformations needed for two such solved systems.

The study of sensitive data, typically shared under nondisclosure agreements, is necessary for maintaining the reliable and secure operation of the electric power grid. However, we must recognize that the development of algorithms and the presentation of results using these models cannot be independently investigated and directly confirmed by others, as is the accepted practice in the scientific community. We need commonly accepted power system models that can be shared broadly, that are accepted as equivalent to actual models that are not shared, and that are suitable for research purposes. This paper shows a method for converting DC OPF models in a way that preserves the confidentiality of the original model and the structure of a DC OPF. This work may serve to more freely allow sharing of realistic models among researchers and thereby aid the process of algorithmic development for solving OPF problems.

APPENDIX
PLOTTING FEASIBLE SPACE

In this Appendix, the procedure for reducing a quadratic or linear program by eliminating equality constraints is detailed. Additionally, the cost function is reconstructed for the reduced system, which can allow plotting of feasible spaces for small-scale problems.

Consider a quadratic program such as (12) with equality constraints $\boldsymbol{M}x = b$, where $\boldsymbol{M}$ has m rows and n columns and $m \leq n$. All m equality constraints and m of the optimization variables in x can be eliminated from the problem. Denote $\boldsymbol{M} = [\boldsymbol{M_1}\ \ \boldsymbol{M_2}]$, where $\boldsymbol{M_1}$ is $m \times (n - m)$ and $\boldsymbol{M_2}$ is $m \times m$; also denote $x = [x_1^T\ \ x_2^T]^T$ where x_1 is $(n - m) \times 1$ and x_2 is $m \times 1$. The optimization variables x_2 can be eliminated from the problem, shown in (28).

$$x_2(x_1) = \boldsymbol{M_2}^{-1}(b - \boldsymbol{M_1} x_1) \tag{28}$$

In (28), x_2 becomes a function of the remaining variables x_1. In a DC OPF problem, x_1 contains $n_{gen} - 1$ optimization variables. One choice of variables in x_1 and x_2 is shown below.

$$x_1 = \left[\, P_{g,1}\ \ P_{g,2}\ \ \ldots\ \ P_{g,n_{gen}-1} \,\right]^T$$
$$x_2(x_1) = \left[\, P_{g,slack}(x_1)\ \ \delta(x_1)^T\ \ x_{sl}(x_1)^T \,\right]^T$$

In other words, the power output by all generators, excluding the slack generator, determines the bus angles δ and operating point of the system. The quadratic cost matrix $\boldsymbol{D}$ and linear cost vector c in (12) can be similarly split into parts. In the case of linear programs, assume $\boldsymbol{D} = \boldsymbol{0}$ in the following derivation.

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{D_{11}} & \boldsymbol{D_{12}} \\ \boldsymbol{D_{21}} & \boldsymbol{D_{22}} \end{bmatrix} \text{ and } c = [c_1^T\ \ c_2^T]^T$$

With optimization variables x_2 eliminated, consider the reduced sized quadratic cost matrix $\widehat{\boldsymbol{D}}$ and linear cost vector $\hat{c}$.

$$\begin{aligned} \widehat{\boldsymbol{D}} = {} & \boldsymbol{M_1^T} {\boldsymbol{M_2^{-1}}}^T \boldsymbol{D_{22}} \boldsymbol{M_2^{-1}} \boldsymbol{M_1} \\ & - \boldsymbol{M_1^T} {\boldsymbol{M_2^{-1}}}^T \boldsymbol{D_{21}} - \boldsymbol{D_{12}} \boldsymbol{M_2^{-1}} \boldsymbol{M_1} + \boldsymbol{D_{11}} \end{aligned}$$

$$\begin{aligned} \hat{c} = {} & c_1^T - c_2^T \boldsymbol{M_2^{-1}} \boldsymbol{M_1} - \tfrac{1}{2} b^T {\boldsymbol{M_2^{-1}}}^T (\boldsymbol{D_{22}} + \boldsymbol{D_{22}^T}) \boldsymbol{M_2^{-1}} \boldsymbol{M_1} \\ & + \tfrac{1}{2} b^T {\boldsymbol{M_2^{-1}}}^T \boldsymbol{D_{12}^T} + \tfrac{1}{2} b^T {\boldsymbol{M_2^{-1}}}^T \boldsymbol{D_{12}^T} \end{aligned}$$

The quadratic program in (12) can be rewritten once more as shown in (29).

$$\begin{aligned} \min_{x_1}\ & \tfrac{1}{2} x_1^T \widehat{\boldsymbol{D}} x_1 + \hat{c}^T x_1 \\ s.t.\ \ & x_2(x_1) \geq 0 \end{aligned} \tag{29}$$

The problem's feasible space in (29) is now clearly determined by $x_2(x_1) \geq 0$. If there are only two or three variables in x_1, then the feasible space can be plotted and visualized in two or three dimensionally respectively.

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