Incorporating Squirrel-Cage Induction Machine Models in Convex Relaxations of OPF Problems

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***Abstract*—The optimal power flow (OPF) problem determines a minimum cost operating point for an electric power system. Recently developed convex relaxations are capable of globally solving certain OPF problems. Using a semidefinite relaxation of the OPF problem as an illustrative example, this letter presents a method for extending convex relaxations of the OPF problem to include steady-state squirrel-cage induction machine models.**

I. Introduction

Operating points for electric power systems are obtained by solving optimal power flow (OPF) problems. Equality constraints in an OPF problem are dictated by the network physics (i.e., the power flow equations) and inequality constraints are determined by engineering limits.

The OPF problem is non-convex and strongly NP-Hard [1]. Recent work (e.g., [2]–[4]) has developed convex relaxations that lower bound the optimal objective value, can certify infeasibility, and, in some cases, provide the global optimum.

Load models are a key component of OPF problems. Many loads are appropriately represented using the "ZIP" model, which is comprised of constant impedance, constant current, and constant power components. An approximation of the ZIP model can be incorporated into OPF relaxations [5].

This letter proposes a method for modeling another class of loads in convex relaxations of OPF problems: steady-state models of squirrel-cage induction machines with specified active power demand.[1] Induction machines are often components of load models [6] and thus relevant to both power system optimization and initialization of dynamic simulations [7], [8].

This letter is organized as follows. Section II presents the OPF problem and a semidefinite programming (SDP) relaxation. Section III describes the steady-state induction machine model and the proposed method for incorporating it into convex relaxations of OPF problems. Section IV presents results for a small test case.

II. Optimal Power Flow and an SDP Relaxation

This section presents the OPF problem and an SDP relaxation. Consider an n-bus system, where $\mathcal{N} = \{1,\ldots,n\}$ is the set of buses, $\mathcal{G}$ is the set of generator buses, and $\mathcal{L}$ is the set of lines. Let $\mathbf{Y}$ denote the network admittance matrix. Using a constant power load model, let $P_{Dk} + \mathbf{j}Q_{Dk}$ represent the active and reactive load demand at bus $k \in \mathcal{N}$, where $\mathbf{j}$ is the imaginary unit. Let V_k represent the voltage phasor at bus $k \in \mathcal{N}$, with the angle of V_1 equal to zero to set the angle reference. Define the rank-one matrix $\mathbf{W} = VV^H \in \mathbb{H}^n$, where $\mathbb{H}^n$ denotes the set of $n \times n$ Hermitian matrices. Superscripts "max" and "min" denote specified upper and lower limits. Buses without generators have maximum and minimum generation set to zero. Define a convex quadratic cost of active power generation with coefficients $c_{2,k} \geq 0$, $c_{1,k}$, and $c_{0,k}$ for $k \in \mathcal{G}$.

Each line $(l,m) \in \mathcal{L}$ is modeled by an ideal transformer with turns ratio $\tau_{lm} e^{\mathbf{j}\theta_{lm}} : 1$ in series with a Π circuit with mutual admittance y_{lm} and total shunt susceptance $\mathbf{j}b_{sh,lm}$. Define e_k as the k^{th} column of the identity matrix. Let $\overline{(\cdot)}$, $(\cdot)^\intercal$, and $(\cdot)^H$ denote the complex conjugate, transpose, and complex conjugate transpose, respectively. Define the matrices $\mathbf{H}_k = \frac{\mathbf{Y}^H e_k e_k^\intercal + e_k e_k^\intercal \mathbf{Y}}{2}$, $\tilde{\mathbf{H}}_k = \frac{\mathbf{Y}^H e_k e_k^\intercal - e_k e_k^\intercal \mathbf{Y}}{2\mathbf{j}}$, $\mathbf{F}_{lm} = \frac{1}{\tau_{lm}^2}\left(\overline{y}_{lm} - \mathbf{j}b_{sh,lm}/2\right) e_l e_l^\intercal - \overline{y}_{lm} / \left(\tau_{lm} e^{-\mathbf{j}\theta_{lm}}\right) e_m e_l^\intercal$, and $\mathbf{F}_{ml} = \left(\overline{y}_{lm} - \mathbf{j}b_{sh,lm}/2\right) e_m e_m^\intercal - \overline{y}_{lm} / \left(\tau_{lm} e^{\mathbf{j}\theta_{lm}}\right) e_l e_m^\intercal$.

The OPF problem is

$$\min_{V \in \mathbb{C}^n,\, \alpha \in \mathbb{R}^{|\mathcal{G}|}} \sum_{k \in \mathcal{G}} \alpha_k \qquad \text{s.t.} \tag{1a}$$

$$P_k^{min} \leq \text{tr}\left(\mathbf{H}_k \mathbf{W}\right) + P_{Dk} \leq P_k^{max} \qquad \forall k \in \mathcal{N} \tag{1b}$$

$$Q_k^{min} \leq \text{tr}(\tilde{\mathbf{H}}_k \mathbf{W}) + Q_{Dk} \leq Q_k^{max} \qquad \forall k \in \mathcal{N} \tag{1c}$$

$$\left(V_k^{min}\right)^2 \leq \text{tr}\left(e_k e_k^\intercal \mathbf{W}\right) \leq \left(V_k^{max}\right)^2 \qquad \forall k \in \mathcal{N} \tag{1d}$$

$$c_{2,k}\left(\text{tr}\left(\mathbf{H}_k \mathbf{W}\right) + P_{Dk}\right)^2 + c_{1,k}\left(\text{tr}\left(\mathbf{H}_k \mathbf{W}\right) + P_{Dk}\right) + c_{0,k} \leq \alpha_k \qquad \forall k \in \mathcal{G} \tag{1e}$$

$$\left\{\text{tr}\left[\left(\mathbf{F}_{lm} + \mathbf{F}_{lm}^H\right)\mathbf{W}\right]\right\}^2 + \left\{\text{tr}\left[\mathbf{j}\left(\mathbf{F}_{lm}^H - \mathbf{F}_{lm}\right)\mathbf{W}\right]\right\}^2 \leq 4\left(S_{lm}^{max}\right)^2 \qquad \forall\,(l,m) \in \mathcal{L} \tag{1f}$$

$$\left\{\text{tr}\left[\left(\mathbf{F}_{ml} + \mathbf{F}_{ml}^H\right)\mathbf{W}\right]\right\}^2 + \left\{\text{tr}\left[\mathbf{j}\left(\mathbf{F}_{ml}^H - \mathbf{F}_{ml}\right)\mathbf{W}\right]\right\}^2 \leq 4\left(S_{lm}^{max}\right)^2 \qquad \forall\,(l,m)\, \mathcal{L} \tag{1g}$$

$$\mathbf{W} = VV^H \tag{1h}$$

where $\text{tr}\left(\cdot\right)$ is the trace and α_k is an auxiliary variable for the objective function of generator $k \in \mathcal{G}$. Constraints on active power (1b), reactive power (1c), and squared voltage magnitude (1d) are linear in the entries of $\mathbf{W}$. Constraints associated with quadratic generation cost (1e) and line flows (1f)–(1g) have a convex second-order cone formulation. Thus, all the non-convexity in (1) is contained in the rank constraint (1h).

An SDP relaxation of the OPF problem (1) is formed by replacing (1h) with a positive semidefinite constraint [2]:

$$\min_{\mathbf{W} \in \mathbb{H}^n, \alpha \in \mathbb{R}^{|\mathcal{G}|}} \sum_{k \in \mathcal{G}} \alpha_k \quad \text{s.t.} \quad \text{(1b)–(1g)}, \quad \mathbf{W} \succeq 0. \tag{2}$$

If the condition $\text{rank}\left(\mathbf{W}\right) = 1$ is satisfied, the SDP relaxation is *exact*. The globally optimal voltages are $V^\star = \sqrt{\lambda}\,\eta$, where λ is the non-zero eigenvalue of $\mathbf{W}$ with associated unit-length eigenvector η, rotated so that the angle of η_1 equals zero.

III. Incorporating Induction Machines Models

The OPF formulation (1) uses a constant power load model. The steady-state equivalent circuit of a single-cage induction

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[1] Induction machines with specified mechanical loads are not considered.

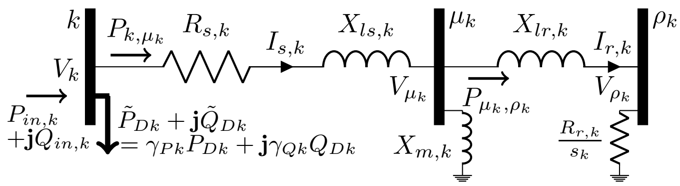

Fig. 1. Induction Machine Steady-State Equivalent Circuit at Bus k

machine, shown in Fig. 1, does not exhibit a constant power characteristic. Rather, the active and reactive demands and voltage magnitude are coupled through a non-linear circuit that depends on the induction machine's slip s. Typically, a specified percentage (here denoted $(1-\gamma_{Pk})$) of the total active power demand P_{Dk} is attributed to induction machine load at bus k and a terminal voltage magnitude $|V_k|$ is specified. Algorithms for solving non-linear equations are used to compute the slip, reactive power demand, and internal currents corresponding to the specified active power and voltage magnitude [7], [8]. This approach is difficult to directly implement in optimization algorithms where voltage magnitudes are decision variables.

This section proposes a method for incorporating the induction machine model in Fig. 1 into the SDP relaxation of the OPF problem (2).[2] For an induction machine at bus k, add two internal buses, μ_k and ρ_k, to the network with new voltage variables V_{μ_k} and V_{ρ_k}. Bus k is connected to bus μ_k through the machine's stator resistance $R_{s,k}$ and leakage reactance $X_{ls,k}$. Bus μ_k is connected to bus ρ_k through the rotor leakage reactance $X_{lr,k}$. All values are in per unit on the system base.

The induction machine's active power demand, P_{k,μ_k}, is a specified quantity $(1-\gamma_{Pk})\,P_{Dk}$. This constrains the active power flowing from bus k to bus μ_k:

$$P_{k,\mu_k} = \mathrm{tr}\left[\frac{1}{2}\left(\mathbf{F}_{k,\mu_k} + \mathbf{F}_{k,\mu_k}^{H}\right)\mathbf{W}\right] = (1-\gamma_{Pk})\,P_{Dk}. \quad (3)$$

The demand at bus k is determined by the load not attributed to the induction machine (i.e., $\tilde{P}_{Dk} + \mathbf{j}\tilde{Q}_{Dk} = \gamma_{Pk}P_{Dk} + \mathbf{j}\gamma_{Qk}Q_{Dk}$, where γ_{Qk} is the percent of Q_{Dk} not attributed to the induction machine, and $P_k^{max} = P_k^{min} = Q_k^{max} = Q_k^{min} = 0$).[3] Thus, the total active power flow into bus k, $P_{in,k}$, is equal to the originally specified active power demand P_{Dk}, while the total reactive power flow into bus k, $Q_{in,k}$, is a function of the induction machine's reactive demand. Bus μ_k has zero power demand (i.e., $P_{\mu_k}^{max} = P_{\mu_k}^{min} = Q_{\mu_k}^{max} = Q_{\mu_k}^{min} = P_{D,\mu_k} = Q_{D,\mu_k} = 0$) and the induction machine's mutual reactance X_m as a shunt, which modifies the admittance matrix entry $\mathbf{Y}_{\mu_k,\mu_k}$. No reactive power is consumed at bus ρ_k, so $Q_{\rho_k}^{max} = Q_{\rho_k}^{min} = Q_{D,\rho_k} = 0$. The active power flowing into ρ_k is allowed to vary such that the terminal active power constraint (3) is satisfied (i.e., $P_{\rho_k}^{\max} = \infty$ and $P_{\rho_k}^{min} = -\infty$); no shunt elements or loads are explicitly modeled at bus ρ_k. The internal buses have no voltage magnitude limits (i.e., $\left(V_{\mu_k}^{min}\right) = \left(V_{\rho_k}^{min}\right) = 0$ and $\left(V_{\mu_k}^{max}\right) = \left(V_{\rho_k}^{max}\right) = \infty$).

If the SDP relaxation (2) for the modified OPF problem has a solution that satisfies $\mathrm{rank}\,(\mathbf{W}) = 1$, globally optimal values for the induction machine's internal variables can be recovered using $V^\star = \sqrt{\lambda}\,\eta$. The stator and rotor currents for the induction machine at bus k, $I_{s,k}^\star$ and $I_{r,k}^\star$, are

$$I_{s,k}^\star = \frac{V_k^\star - V_{\mu_k}^\star}{R_{s,k} + \mathbf{j}X_{ls,k}}, \qquad I_{r,k}^\star = \frac{V_{\mu_k}^\star - V_{\rho_k}^\star}{\mathbf{j}X_{lr,k}}. \quad (4)$$

The induction machine's slip is

$$s_k^\star = R_{r,k}\frac{V_{\rho_k}^\star \overline{I}_{r,k}^\star + \overline{V}_{\rho_k}^\star I_{r,k}^\star}{2\left|V_{\rho_k}^\star\right|^2}. \quad (5)$$

Incorporating induction machine models can affect a relaxation's exactness. Tighter relaxations, such as higher-order "moment" relaxations [4], may be applied if $\mathrm{rank}\,(\mathbf{W}) > 1$.

Note that the proposed approach can also be applied to power flow problems by appropriate choice of the OPF limits.

[2]The proposed induction machine modeling approach is also applicable to other relaxations (e.g., [3], [4]) and can be extended to double-cage models.

[3]Other load models, e.g., ZIP, can alternatively be used for this demand.

IV. Numerical Test Case

The proposed approach is illustrated using the IEEE 14-bus OPF problem [9], modified to include two aggregate induction machine models with the parameters in Table I.

TABLE I
INDUCTION MACHINE PARAMETERS (PER UNIT)

Bus	γ_{Pk}	γ_{Qk}	X_{ls}	X_{lr}	X_m	R_s	R_r
4	0.0	0.0	0.07	0.17	3.5	0.012	0.010
9	0.0	0.0	0.23	0.23	5.8	0.001	0.015

The SDP relaxation is exact for this problem ($\mathrm{rank}\,(\mathbf{W}) = 1$) which enables computation of $V^\star$ using an eigendecomposition and $I_s^\star$, $I_r^\star$ and $s^\star$ using (4) and (5) as given in Table II.

TABLE II
GLOBAL OPTIMUM FOR INDUCTION MACHINE VARIABLES (PER UNIT)

Bus	$V_k^\star$	$V_{\mu_k}^\star$	$V_{\rho_k}^\star$	$I_{s,k}^\star$	$I_{r,k}^\star$	$s_k^\star$
13	0.998 $\angle$-8.4°	0.969 $\angle$-10.2°	0.966 $\angle$-15.1°	0.584 $\angle$-43.3°	0.491 $\angle$-15.1°	0.0051
14	1.031 $\angle$-12.8°	0.985 $\angle$-16.6°	0.983 $\angle$-20.6°	0.355 $\angle$-49.1°	0.300 $\angle$-20.6°	0.0046

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