POWER SYSTEM MODELS FORMULATED AS EIGENVALUE PROBLEMS AND PROPERTIES OF THEIR SOLUTIONS

by

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ABSTRACT

Eigenvalue problems incorporate multiplicative nonlinearities in otherwise linear equations. Two power systems models with multiplicative nonlinearities are reformulated as eigenvalue problems. Reformulation allows for the application of eigenvalue theory and solution techniques to these models.

The first model involves determination of the initial conditions for induction machine internal variables in a non-linear dynamic power system analysis. The internal variables of stator and rotor currents, rotor speed, and mechanical torque must be determined from the given values of input real power, stator voltage magnitude, and stator voltage angle. This problem is posed in an eigenvalue formulation that can be solved using standard linear algebra techniques. The eigenvalue formulation allows for determination of all possible solutions for the internal variables rather than the single solution obtained from traditional iterative methods. Additionally, the absence of non-zero real eigenvalues indicates that the problem has no solution. Both single-cage and double-cage induction machines are considered, and numeric examples are presented.

The second model involves reformulation of the power flow equations as a multiparameter eigenvalue problem. In this formulation, both the eigenvalues and the eigenvectors are composed of the d and q orthogonal components of the bus voltages. The two parameter formulation of the power flow equations for two bus systems can be solved directly by decomposing the problem into two generalized eigenvalue problems that must be simultaneously satisfied. Since n bus systems require $2\,(n-1)$ parameter eigenvalue problems, which do not yet have a general solution method for $n > 2$, systems with more than two buses are not yet directly solvable from the multiparameter eigenvalue formulation.

In addition to direct solution, two other research avenues for the multiparameter eigenvalue formulation of the power flow equations are pursued. First, motivated by eigenvalue problem structure, a reformulation of the multiparameter eigenvalue form of the power flow equations is presented. Although it is without known practical applications, this reformulation and the intermediate results in its derivation are interesting from a theoretical standpoint. Second, an eigenvalue sensitivity analysis is performed on the multiparameter eigenvalue formulation of the power flow equations. The linearization obtained from the eigenvalue sensitivity analysis is equivalent to the linearization obtained from the power flow Jacobian.

Future developments in multiparameter eigenvalue theory may provide additional insights into solutions of the power flow equations. General solution techniques for multiparameter eigenvalue problems with more than two parameters may enable direct solution of the power flow equations. A method for determining the number of solutions to multiparameter eigenvalue problems would be useful as a stopping condition for continuation power flows. Conditions for the existence of any solutions to multiparameter eigenvalue problems would be useful for finding the point of voltage collapse and for analyzing power systems in heavily loaded conditions.

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Introduction

Eigenvalue problems incorporate multiplicative nonlinearities in otherwise linear equations. The standard eigenvalue problem, $\mathbf{A}x = \lambda x$, has a single multiplicative nonlinearity, namely, multiplication of the eigenvector x by the eigenvalue λ. A multiparameter eigenvalue problem with n parameters incorporates n multiplicative nonlinearities. Eigenvalue formulations of engineering problems are theoretically interesting and can also be practically useful. Eigenvalue formulations have significant advantages due to the large established body of eigenvalue research and the potential application of future mathematical developments in eigenvalue theory.

This thesis investigates eigenvalue formulations of two models used in the field of electrical power systems: the induction machine initial conditions problem and the power flow equations. This introduction provides a discussion of relevant eigenvalue theory and literature as well as an overview of the two power systems models that are investigated in this thesis.

1.1 Eigenvalue Problems

There are three eigenvalue problems that will be used in the rest of this thesis: the standard eigenvalue problem, the generalized eigenvalue problem, and the multiparameter eigenvalue problem. The standard and generalized eigenvalue problems will be used in the induction machine initial conditions chapter and all three types of eigenvalue problems will be used in the power flow equations chapter.

This section provides a brief overview of each eigenvalue problem. For a more detailed treatment of the standard and generalized eigenvalue problems, see [10].

1.1.1 The Standard Eigenvalue Problem

The standard eigenvalue problem is commonly used in engineering and mathematics. It describes a scalar known as the eigenvalue and a vector known as the eigenvector that are associated with a matrix. Two types of standard eigenvalue problems are possible: right and left. A right eigenvalue problem is defined in (1.1), where λ is the eigenvalue, x is a right eigenvector, and $\mathbf{A}$ is a matrix.

$$\mathbf{A}x = \lambda x \tag{1.1}$$

A left eigenvalue problem is defined in (1.2), where λ is the eigenvalue, y is a left eigenvector, superscript T indicates the transpose operation, and $\mathbf{A}$ is a matrix.

$$y^T\mathbf{A} = \lambda y^T \tag{1.2}$$

Matrix $\mathbf{A}$ represents a linear map from one vector space (the domain) to another vector space (the range). While a general vector in the domain is both scaled and rotated when mapped to the range vector space, eigenvectors are only scaled. The amount of the scaling is determined by the eigenvalue associated with that eigenvector. For example, if the matrix $\mathbf{A}$ has an eigenvector x associated with eigenvalue λ, then mapping the eigenvector from the domain vector space to the range vector space $(\mathbf{A}x)$ will yield a parallel vector that is scaled by the eigenvalue (λx).

The standard eigenvalue problem can solved by finding the roots of the associated characteristic polynomial. The eigenvalue problem can be rearranged as $(\mathbf{A} - \lambda\mathbf{I})\,x = 0$, where $\mathbf{I}$ is the suitably sized identity matrix. If $(\mathbf{A} - \lambda\mathbf{I})$ were invertible, the eigenvector x would be the trivial solution of the zero vector. To obtain a non-trivial x, $(\mathbf{A} - \lambda\mathbf{I})$ must not be invertible, and thus have zero determinant. This gives the characteristic equation

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0 \tag{1.3}$$

Finding all solutions to (1.3) yields all of the eigenvalues λ. Since the characteristic polynomial can have a high order, practical eigenvalue codes such as [1] use iterative solution techniques. However, matrices with rank less than or equal to four have characteristic polynomials of order less than or equal to four. Since roots of quartic, cubic, quadratic, and linear polynomials can be directly solved without iteration, eigenvalue problems for matrices of rank four or less do not require iterative solution techniques. See [20] for the methods of finding the roots of quartic, cubic, and quadratic polynomials.

An eigenvector associated with a given eigenvalue λ can then be found as a vector in the nullspace of $(\mathbf{A} - \lambda\mathbf{I})$. Eigenvectors have one degree of freedom in their magnitude. In other words, if x is an eigenvector, then ax is also an eigenvector for any non-zero scalar a. Therefore, assuming no repeated eigenvalues, solution of one eigenvector associated with a given eigenvalue allows for easy identification of all possible eigenvectors associated with that eigenvalue.

1.1.2 The Generalized Eigenvalue Problem

The generalized eigenvalue problem introduces a second matrix to the standard eigenvalue problem. (1.4) shows the right generalized eigenvalue problem, where $\mathbf{A}$ and $\mathbf{B}$ are matrices, λ is the eigenvalue and x is a right eigenvector. (1.5) shows the left generalized eigenvalue problem, where $\mathbf{A}$ and $\mathbf{B}$ are matrices, λ is the eigenvalue, and y is a left eigenvector.

$$\mathbf{A}x = \lambda\mathbf{B}x \tag{1.4}$$

$$y^T\mathbf{A} = \lambda y^T\mathbf{B} \tag{1.5}$$

If the matrix $\mathbf{B}$ is invertible, generalized eigenvalue problems can be converted to standard eigenvalue problems. Assuming a non-singular $\mathbf{B}$, (1.4) can be rearranged in the form of (1.1):

$$\mathbf{B}^{-1}\mathbf{A}x = \lambda x \tag{1.6}$$

If the matrix $\mathbf{B}$ is not invertible, the generalized eigenvalue problem can be solved directly using matrix pencil methods [1]. The linear matrix pencil method used for generalized eigenvalue problems solves the equation $\det(\mathbf{A} - \lambda\mathbf{B}) = 0$, similar to the characteristic equation for standard eigenvalue problems (1.3). This allows for the solution of the generalized eigenvalue problem without explicitly forming the matrix $\mathbf{B}^{-1}\mathbf{A}$.

Like the standard eigenvalue problem, an eigenvector for the generalized eigenvalue problem has one degree of freedom in its magnitude. If x is an eigenvector, then ax is also an eigenvector for any non-zero scalar a.

The generalized eigenvalue problem has robust solution techniques and well developed theory. Practical implementations of generalized eigenvalue problem solvers are common. For instance, MATLAB's `EIG` function has the ability to solve generalized eigenvalue problems.

1.1.3 The Multiparameter Eigenvalue Problem

The k-parameter eigenvalue problem combines k eigenvalues and k equations into a single problem. The right k-parameter eigenvalue problem can be represented as in (1.7), where λ_j is a scalar eigenvalue, $\mathbf{V_{ij}}$ is a matrix, and x_i is an eigenvector.

$$\left(\mathbf{V_{i0}} + \sum_{j=1}^{k} \lambda_j \mathbf{V_{ij}}\right) x_i = 0, \quad i = 1, \cdots, k \tag{1.7}$$

The left k-parameter eigenvalue problem can be represented as in (1.8), where y_i is a left eigenvector.

$$y_i^T \left(\mathbf{V_{i0}} + \sum_{j=1}^{k} \lambda_j \mathbf{V_{ij}}\right) = 0, \quad i = 1, \cdots, k \tag{1.8}$$

For ease of notation, the matrix sum in both left and right formulations is written as $\mathbf{W_i}(\lambda)$, where λ indicates the vector $\begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_k\end{bmatrix}^T$.

$$\mathbf{W_i}(\lambda) = \mathbf{V_{i0}} + \sum_{j=1}^{k} \lambda_j \mathbf{V_{ij}}, \quad i = 1, \cdots, k \tag{1.9}$$

To aid in comprehension, the right four parameter eigenvalue problem is shown in (1.10).

$$\begin{aligned}
\left(\mathbf{A_0} + \lambda_1 \mathbf{A_1} + \lambda_2 \mathbf{A_2} + \lambda_3 \mathbf{A_3} + \lambda_4 \mathbf{A_4}\right) x_1 &= 0 \\
\left(\mathbf{B_0} + \lambda_1 \mathbf{B_1} + \lambda_2 \mathbf{B_2} + \lambda_3 \mathbf{A_3} + \lambda_4 \mathbf{B_4}\right) x_2 &= 0 \\
\left(\mathbf{C_0} + \lambda_1 \mathbf{C_1} + \lambda_2 \mathbf{C_2} + \lambda_3 \mathbf{A_3} + \lambda_4 \mathbf{C_4}\right) x_3 &= 0 \\
\left(\mathbf{D_0} + \lambda_1 \mathbf{D_1} + \lambda_2 \mathbf{D_2} + \lambda_3 \mathbf{A_3} + \lambda_4 \mathbf{D_4}\right) x_4 &= 0
\end{aligned} \tag{1.10}$$

The theory of multiparameter eigenvalue problems is not as mature as the theory of standard and generalized eigenvalue problems. Much of multiparameter eigenvalue theory assumes that the $\mathbf{V_{ij}}$ matrices are Hermitian or that the matrix $\mathbf{W_i}(\lambda)$ is left or right definite. For instance, the books [2] and [18] work almost entirely with Hermitian multiparameter eigenvalue problems. Definiteness or Hermitian forms of multiparameter eigenvalue problems arise naturally from the study of physical phenomena such as the wave equation and therefore have a greater theoretical development. The multiparameter eigenvalue formulation of the power flow equations is neither Hermitian nor left or right definite and thus cannot be analyzed with theory developed for specialized forms of multiparameter eigenvalue problems. Two references relevant to this thesis were found that describe interesting results for multiparameter eigenvalue problems without restrictions.

The first reference, [5], describes a variety of methods for solving two parameter eigenvalue problems. Two parameters are sufficient for the power flow equations associated with a two bus power system (since one bus is a slack bus with known values of voltage magnitude and angle). Since the $\mathbf{V_{ij}}$ matrices required in the power flow equation are small (3×3), the Kronecker product method described in Chapter 2 of [5] can be used to solve the power flow equations for two bus systems. The Kronecker product method converts the multiparameter eigenvalue problem into two generalized eigenvalue problems that must be solved simultaneously.

The Kronecker product, denoted by $\otimes$, takes an $m \times n$ matrix $\mathbf{A}$ and a $p \times q$ matrix $\mathbf{B}$ as inputs to form a $mp \times nq$ matrix $\mathbf{A} \otimes \mathbf{B}$.

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\mathbf{B} & \cdots & a_{nn}\mathbf{B} \end{bmatrix} \tag{1.11}$$

The derivation of the Kronecker product method provided in [5] can be performed using the linearity, distributivity, and mixed product properties of the Kronecker product illustrated in (1.12), (1.13), (1.14), and (1.15), where k is a scalar and $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are matrices.

$$(k\mathbf{A}) \otimes \mathbf{B} = \mathbf{A} \otimes (k\mathbf{B}) = k\,(\mathbf{A} \otimes \mathbf{B}) \tag{1.12}$$

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C} \tag{1.13}$$

$$(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C} \tag{1.14}$$

$$(\mathbf{A} \otimes \mathbf{B})\,(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC} \otimes \mathbf{BD}) \tag{1.15}$$

See [20] for further discussion on the Kronecker product.

Consider the two parameter eigenvalue problem described in (1.16) and (1.17), where λ_1 and λ_2 are eigenvalues and x_1 and x_2 are eigenvectors.

$$\mathbf{A_0}x_1 + \lambda_1\mathbf{A_1}x_1 + \lambda_2\mathbf{A_2}x_1 = 0 \tag{1.16}$$

$$\mathbf{B_0}x_2 + \lambda_1\mathbf{B_1}x_2 + \lambda_2\mathbf{B_2}x_2 = 0 \tag{1.17}$$

To derive the Kronecker product method, λ_1 must be isolated in a generalized eigenvalue problem. Take the Kronecker product of (1.16) with $-\mathbf{B_2}x_2$ and the Kronecker product of $-\mathbf{A_2}x_1$ with (1.17).

$$(\mathbf{A_0}x_1 + \lambda_1\mathbf{A_1}x_1 + \lambda_2\mathbf{A_2}x_1)\otimes(-\mathbf{B_2}x_2) = (-\mathbf{A_2}x_1)\otimes(\mathbf{B_0}x_2 + \lambda_1\mathbf{B_1}x_2 + \lambda_2\mathbf{B_2}x_2) \quad (1.18)$$

Add $\lambda_2\mathbf{A_2}x_1 \otimes \mathbf{B_2}x_2$ to both sides to eliminate the λ_2 terms.

$$(\mathbf{A_0}x_1 + \lambda_1\mathbf{A_1}x_1) \otimes (-\mathbf{B_2}x_2) = (-\mathbf{A_2}x_1) \otimes (\mathbf{B_0}x_2 + \lambda_1\mathbf{B_1}x_2) \quad (1.19)$$

Expand and rearrange this equation.

$$-\mathbf{A_0}x_1 \otimes \mathbf{B_2}x_2 + \mathbf{A_2}x_1 \otimes \mathbf{B_0}x_2 = \lambda_1 (\mathbf{A_1}x_1 \otimes \mathbf{B_2}x_2 - \mathbf{A_2}x_1 \otimes \mathbf{B_1}x_2) \quad (1.20)$$

Then use the mixed product property (1.15).

$$(\mathbf{A_2} \otimes \mathbf{B_0} - \mathbf{A_0} \otimes \mathbf{B_2})(x_1 \otimes x_2) = \lambda_1 (\mathbf{A_1} \otimes \mathbf{B_2} - \mathbf{A_2} \otimes \mathbf{B_1})(x_1 \otimes x_2) \quad (1.21)$$

Similarly, λ_2 must be isolated in a generalized eigenvalue problem. Take the Kronecker product of (1.16) with $-\mathbf{B_1}x_2$ and the Kronecker product of $-\mathbf{A_1}x_1$ with (1.17). Equate the results.

$$(\mathbf{A_0}x_1 + \lambda_1\mathbf{A_1}x_1 + \lambda_2\mathbf{A_2}x_1)\otimes(-\mathbf{B_1}x_2) = (-\mathbf{A_1}x_1)\otimes(\mathbf{B_0}x_2 + \lambda_1\mathbf{B_1}x_2 + \lambda_2\mathbf{B_2}x_2) \quad (1.22)$$

Add $\lambda_1\mathbf{A_1}x_1 \otimes \mathbf{B_1}x_2$ to both sides to eliminate the λ_1 terms.

$$(\mathbf{A_0}x_1 + \lambda_2\mathbf{A_2}x_1) \otimes (-\mathbf{B_1}x_2) = (-\mathbf{A_1}x_1) \otimes (\mathbf{B_0}x_2 + \lambda_2\mathbf{B_2}x_2) \quad (1.23)$$

Expand and rearrange this equation.

$$\mathbf{A_0}x_1 \otimes \mathbf{B_1}x_2 - \mathbf{A_1}x_1 \otimes \mathbf{B_0}x_2 = \lambda_2 \left(\mathbf{A_1}x_1 \otimes \mathbf{B_2}x_2 - \mathbf{A_2}x_1 \otimes \mathbf{B_1}x_2\right) \tag{1.24}$$

Then use the mixed product property (1.15).

$$\left(\mathbf{A_0} \otimes \mathbf{B_1} - \mathbf{A_1} \otimes \mathbf{B_0}\right)\left(x_1 \otimes x_2\right) = \lambda_2 \left(\mathbf{A_1} \otimes \mathbf{B_2} - \mathbf{A_2} \otimes \mathbf{B_1}\right)\left(x_1 \otimes x_2\right) \tag{1.25}$$

(1.21) and (1.25) are generalized eigenvalue problems

$$\mathbf{\Delta_1} z = \lambda_1 \mathbf{\Delta_0} z \tag{1.26}$$

$$\mathbf{\Delta_2} z = \lambda_2 \mathbf{\Delta_0} z \tag{1.27}$$

where

$$\mathbf{\Delta_0} = \mathbf{A_1} \otimes \mathbf{B_2} - \mathbf{A_2} \otimes \mathbf{B_1} \tag{1.28}$$

$$\mathbf{\Delta_1} = \mathbf{A_2} \otimes \mathbf{B_0} - \mathbf{A_0} \otimes \mathbf{B_2} \tag{1.29}$$

$$\mathbf{\Delta_2} = \mathbf{A_0} \otimes \mathbf{B_1} - \mathbf{A_1} \otimes \mathbf{B_0} \tag{1.30}$$

$$z = x_1 \otimes x_2 \tag{1.31}$$

The solution to the two-parameter eigenvalue problem can be obtained from a simultaneous solution of (1.26) and (1.27).

Since each bus besides the slack bus has two degrees of freedom, the number of parameters necessary to represent an n-bus power system is $2\left(n-1\right)$. It is not clear how the Kronecker product method (or any other methods described in [5]) can be applied to multiparameter eigenvalue problems with more than two parameters. In fact, no solution techniques for multiparameter eigenvalue problems with more than two parameters were found in any existing literature. Solving multiparameter eigenvalue problems that have more than two parameters is believed to be an open problem. Therefore, new developments in multiparameter eigenvalue theory will be needed before direct solution of the multiparmeter eigenvalue formulation of the power flow equations will be possible for practical size power systems.

The second reference, [11], investigates condition numbers for multiparameter eigenvalue problems. Condition numbers indicate the largest variation of an output that can occur due to variation of an input or parameter. [11] derives the condition number for the eigenvalues of the multiparameter eigenvalue problem as a function of variation of the matrices $\mathbf{V_{ij}}$. While the condition number itself is not of primary interest to this thesis, an intermediate result of the eigenvalue condition number derivation in the paper is useful; namely, the sensitivity of the eigenvalues to variations in the matrices $\mathbf{V_{ij}}$. This sensitivity is obtained by first multiplying the right multiparameter eigenvalue equation (1.7) by the corresponding left eigenvectors y^* (in the power flow equation, as will be shown in Section 3.3, all eigenvectors are real so the complex conjugation indicated by y^* is equivalent to the transpose y^T). Next, all eigenvalues, eigenvectors, and matrices in the resulting equation are perturbed by an amount Δ. The perturbed equation is expanded and simplified by discarding all second order and higher terms. The result can be reduced to the form shown in (1.32).

$$\begin{bmatrix} y_1^* \mathbf{V_{11}} x_1 & \cdots & y_1^* \mathbf{V_{1k}} x_1 \\ \vdots & & \vdots \\ y_k^* \mathbf{V_{k1}} x_k & \cdots & y_k^* \mathbf{V_{kk}} x_k \end{bmatrix} \begin{bmatrix} \Delta\lambda_1 \\ \vdots \\ \Delta\lambda_k \end{bmatrix} = \begin{bmatrix} y_1^* \Delta \mathbf{W_1} (\lambda) x_1 \\ \vdots \\ y_k^* \Delta \mathbf{W_k} (\lambda) x_k \end{bmatrix} \tag{1.32}$$

In Section 3.6, the eigenvalue sensitivity equation (1.32) will be used to analyze a multiparameter eigenvalue formulation of the power flow equations.

1.2 Induction Machine Initial Conditions Problem Overview

The first model that will be investigated in this thesis involves induction machines. Induction machines are of great importance to power system engineers due to both their prevalence and size. Induction machines are the most common electric machine in most power systems and are often used in both consumer applications such as air conditioners and refrigerators and industrial applications such as pumps and fans. Industrial induction motors can be in the megawatt size range. Hence, induction machines often comprise a substantial percentage of the load at many buses in a power system and can have a large impact on power system operations.

Power system engineers must investigate the dynamic operation of the electric grid in order to ensure transient stability. Engineers use dynamic power system simulations to protect power systems from transients such as faults and lightning strikes and thus maintain transient stability. Since large induction machines can have a substantial effect on power system dynamics, it is important that these machines are properly modeled in dynamic power system simulations. In order to model an induction machine in these simulations, the initial conditions for the machine's rotor and stator currents, rotor speed, and mechanical torque must be determined. The initial conditions are the values of the the currents, speed, and torque immediately before the transient occurs and can be found from the steady state operating point of the induction machine. The eigenvalue formulation for this problem described in Section 2.3 allows for direct solution of all possible sets of initial conditions and can be used to easily determine when no solutions exist.

The traditional method for solving the induction machine initial conditions problem uses iteration on the machine's rotor speed. A disadvantage of this method is that only one solution is obtained when multiple solutions are possible. The iterative method is used in such software as Electro-Magnetic Transients Program (EMTP) [7, pp. 9-19], Positive Sequence Load Flow Software (PSLF) [8, p. 785] and Power System Simulation for Engineering (PSS/E) [17, pp. 20-16].

1.3 Power Flow Equations Overview

The second model that will be investigated in this thesis involves the power flow equations (also known as the load flow equations). These equations use the electrical properties of the transmission network to relate the real and reactive power injected at each bus to the voltage magnitude and voltage angle at each bus in a power system. Since knowledge of the power injections and voltage are essential for successful operation of the electric grid, the power flow equations are extremely important to power system engineers.

Due to the importance of the power flow equations, there has been significant research into solution techniques and characteristics of the solutions. For a textbook treatment of the power flow equations, including their derivation from circuit theory and several standard solution techniques and approximations (Gauss-Seidel, Newton-Raphson, and the DC power flow), see [9]. There has also been significant research attention to the problem of finding all solutions to the power flow equations. [4] uses analytic tools from topology and geometry to determine bounds on the number of solutions and the stability of solutions in a lossless network of PV (specified real power injection and voltage magnitude) buses. [3] generalizes this analysis to lossy systems of PV buses. A continuation power flow algorithm that can reliably find all solutions to the power flow equations was published in [14] and has one implementation described [6]. More esoteric solution techniques for the power flow equations have also been attempted, such as a biologically inspired ant colony algorithm in [12] and a genetic algorithm [19]. None of the existing literature formulates the power flow equations in terms of eigenvalue problems.

In Chapter 3, the power flow equations are formulated in terms of a multiparameter eigenvalue problem. Direct solutions of the multiparameter eigenvalue formulation for a two bus power systems using the Kronecker product method are detailed. In addition to directly searching for the solutions of the power flow equations, the multiparameter eigenvalue formulation provides two other areas of investigation: a further reformulation motivated by standard eigenvalue problem structure and a sensitivity analysis of the multiparameter eigenvalues. Although practical applications of these investigations were not discovered, this thesis details some interesting theoretical results and lays the groundwork for future research into the power flow equations.

Chapter 2

Induction Machine Initial Conditions Problem

2.1 Introduction

Dynamic simulations of power systems generally begin by running a load flow analysis to determine steady state values of the real power (P), the reactive power (Q), the voltage magnitude (V) and the voltage angle (δ) at each bus. Subsequently, the initial conditions for components connected to the bus are set to their steady state value. Induction machines comprise a significant portion of the load at many buses. For each induction machine connected to a bus, the initial conditions of the machine model's internal state variables must be obtained such that the terminal variables of the model match the values of P, Q, V, and δ obtained from the load flow analysis. Appropriate choice of the internal states of the induction machine model can set both the machine's real power and voltage to match the bus values that are obtained from the load flow analysis. An additional capacitor is often added at the bus to ensure that the reactive power output of the machine matches the reactive power obtained from the load flow analysis [16].

A common approach for matching the real power and voltage uses iteration on the induction machine's slip to solve the machine equations for the internal variables using the known terminal values. This approach is used in such software as Electro-Magnetic Transients Program (EMTP) [7, pp. 9-19], Positive Sequence Load Flow Software (PSLF) [8, p. 785] and Power System Simulation for Engineering (PSS/E) [17, pp. 20-16]. A potential disadvantage of the iterative approach is that only one solution is obtained when multiple solutions are often possible. For general purposes the highest speed stable solution is sought; however, for some research purposes, studies can focus on other unstable solutions.

This section presents an eigenvalue approach for solving the induction machine initial conditions problem. Assuming linear magnetic relationships, the electrical equations appear nearly linear, with non-linearity involving a common multiplier, the rotor speed. This will serve as the eigenvalue in our formulation. The eigenvector is composed of the machine electrical variables. Once these are solved, the torque equation is used to initialize the model's mechanical torque.

This eigenvalue approach has the advantage of providing all solutions, stable and unstable, and can reliably determine when no solution exists through the absence of non-zero real eigenvalues.

This chapter first describes a single-cage dynamic induction machine model and derives the eigenvalue formulation for the induction machine initial conditions problem. Next, the eigenvalue formulation is extended to a double-cage induction machine model. Numeric examples are provided.

2.2 Dynamic Single-Cage Induction Machine Model

This section presents a standard dynamic induction machine model in the dq frame with linear magnetic relationships and short circuited rotor windings. The model was adapted from [13, pp. 284].

$$\begin{bmatrix} V_{ds} \\ V_{qs} \\ 0 \\ 0 \end{bmatrix} = \left(\begin{bmatrix} R_s & -\left(X_{ls}+X_m\right) & 0 & -X_m \\ \left(X_{ls}+X_m\right) & R_s & X_m & 0 \\ 0 & -X_m & R_r & -\left(X_{lr}+X_m\right) \\ X_m & 0 & \left(X_{lr}+X_m\right) & R_r \end{bmatrix} \right.$$
$$\left. + \frac{\omega_r}{\omega_s} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & X_m & 0 & \left(X_{lr}+X_m\right) \\ -X_m & 0 & -\left(X_{lr}+X_m\right) & 0 \end{bmatrix} \right) \begin{bmatrix} I_{ds} \\ I_{qs} \\ I_{dr} \\ I_{qr} \end{bmatrix} \tag{2.1}$$
$$+ \frac{1}{\omega_s} \begin{bmatrix} \left(X_{ls}+X_m\right) & 0 & X_m & 0 \\ 0 & \left(X_{ls}+X_m\right) & 0 & X_m \\ X_m & 0 & \left(X_{lr}+X_m\right) & 0 \\ 0 & X_m & 0 & \left(X_{lr}+X_m\right) \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I_{ds} \\ I_{qs} \\ I_{dr} \\ I_{qr} \end{bmatrix}$$

$$\frac{2H}{\omega_s} \frac{\mathrm{d}\omega_\mathrm{r}}{\mathrm{d}t} = T_e\left(I_{ds}, I_{qs}, I_{dr}, I_{qr}\right) - T_m \tag{2.2}$$

$$T_e\left(I_{ds}, I_{qs}, I_{dr}, I_{qr}\right) = X_m\left(I_{qs} I_{dr} - I_{ds} I_{qr}\right) \tag{2.3}$$

ω_s refers to the electrical excitation frequency and ω_r refers to the rotor speed. X_{ls} is the stator leakage reactance, X_{lr} is the rotor leakage reactance, X_m is the mutual reactance, R_s is the stator resistance, R_r is the rotor resistance, and H is the inertia constant of the machine and mechanical load. All quantities are in per unit.

The familiar steady state equivalent circuit for the induction machine model from (2.1), (2.2), and (2.3) is presented in Fig. 2.1. The machine slip is $s = \frac{\omega_s - \omega_r}{\omega_s}$, the stator current is $I_s = I_{ds} + jI_{qs}$, the rotor current is $I_r = I_{dr} + jI_{qr}$, and the stator voltage is $V_s = V_{ds} + jV_{qs}$.

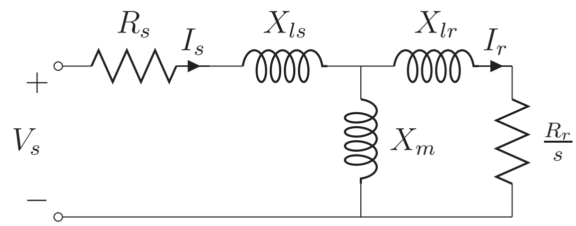

Figure 2.1 Single-Cage Induction Machine Steady State Equivalent Circuit

2.3 Eigenvalue Formulation of the Initial Conditions Problem

The dynamic induction machine model given by (2.1), (2.2), and (2.3) fits into a more general induction machine dynamic model framework.

$$y = [\mathbf{A} + \omega_r \mathbf{B}]\, x + \mathbf{C}\frac{\mathrm{d}x}{\mathrm{dt}} \tag{2.4}$$

$$\frac{2H}{\omega_s}\frac{\mathrm{d}\omega_r}{\mathrm{d}t} = T_e\,(x) - T_m \tag{2.5}$$

This model has the applied voltage contained in the vector y and stator and rotor currents contained in the vector x. The rotor windings are short circuited. $\mathbf{A}$ contains all terms that do not depend on the rotor speed ω_r, $\mathbf{B}$ contains all terms that do depend on ω_r, and $\mathbf{C}$ contains all terms that depend on the derivative of the currents. The model thus has a single multiplicative non-linearity, namely a dependence on ω_r.

In steady state, $\frac{\mathrm{d}\omega_r}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$.

$$y = [\mathbf{A} + \omega_r \mathbf{B}]\, x \tag{2.6}$$

$$T_e\,(x) = T_m \tag{2.7}$$

$\mathbf{A}$ and $\mathbf{B}$ are completely defined by the machine parameters. Since the phase angle of the applied voltage does not affect the power consumption of the motor, we specify the stator d-axis voltage V_{ds} equal to the voltage obtained from the load flow analysis and the stator q-axis voltage V_{qs} equal to zero. The voltage is then directed entirely in the d-axis. This specification can be corrected at the end of the method by rotating the current angles by the bus voltage angle obtained from the load flow analysis. Therefore, the voltage vector y is completely known. Since V_{qs} is specified to be zero, the real power used by the machine is $P = V_{ds}I_{ds}$. The real power is known from the load flow analysis, so the d-axis current can be directly determined.

$$I_{ds} = \frac{P}{V_{ds}} \tag{2.8}$$

The machine model can be put into the form of an eigenvalue problem by combining the known voltage vector and the known matrix $\mathbf{A}$. First rewrite the voltage vector y as

$$y = \begin{bmatrix} V_{ds} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{V_{ds}}{I_{ds}} \\ 0 \\ 0 \\ 0 \end{bmatrix} I_{ds} = \begin{bmatrix} R_A \\ 0 \\ 0 \\ 0 \end{bmatrix} I_{ds} \tag{2.9}$$

where, using (2.8),

$$R_A = \frac{V_{ds}}{I_{ds}} = \frac{V_{in}^2}{P} \tag{2.10}$$

Then define the matrix $\mathbf{D}$ as

$$-\mathbf{D} = -\left[\begin{array}{c|c} R_A & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] + \mathbf{A} \tag{2.11}$$

and rewrite (2.6) as

$$-\mathbf{D}x + \omega_r \mathbf{B}x = 0 \tag{2.12}$$

or, in a well-known generalized eigenvalue form

$$\mathbf{D}x = \omega_r \mathbf{B}x \tag{2.13}$$

This formulation can either be solved with generalized eigenvalue solution techniques or, since $\mathbf{D}$ is invertible, converted to standard eigenvalue form and solved with standard eigenvalue techniques [1].

$$\frac{1}{\omega_r}x = \mathbf{D}^{-1}\mathbf{B}x \tag{2.14}$$

Since the eigenvector can be arbitrarily scaled, rescaling the eigenvector using the known value of I_{ds} from (2.8) is required. Additionally, correction for the voltage angle δ from the load flow analysis is needed: rotate the currents in the vector x by δ after solving the eigenvalue problem. Finally, the mechanical torque is obtained using (2.3) and (2.7).

2.4 Single-Cage Induction Machine Numeric Example

Assume V_{in} is obtained from a load flow analysis. Specify $V_{ds} = V_{in}$ and $V_{qs} = 0$. The method can then be applied to the single-cage induction machine model given by (2.6) and (2.7) as previously described in Section 2.3.

The matrix $\mathbf{D}$ is

$$\mathbf{D} = -\begin{bmatrix} (R_s - R_A) & -(X_{ls} + X_m) & 0 & -X_m \\ (X_{ls} + X_m) & R_s & X_m & 0 \\ 0 & -X_m & R_r & -(X_{lr} + X_m) \\ X_m & 0 & (X_{lr} + X_m) & R_r \end{bmatrix} \tag{2.15}$$

Now ω_r and the currents in x can be found by solving (2.14). Note that the eigenvector x must be scaled correctly. Since the first entry of x should be I_{ds}, scaling can be done by multiplying the each entry in the eigenvector x by I_{ds} divided by the first entry of x.

Consider a single-cage induction machine with the parameter values in per unit representation given in Table 2.1. Assume that the values in Table 2.2 are obtained from a load flow analysis.

X_{ls}	X_{lr}	X_m	R_s	R_r	ω_s (rad/sec)
0.10	0.10	3.5	0.013	0.015	377

Table 2.1 Single-Cage Induction Machine Equivalent Circuit Parameters

P	V_{in}	δ
1.0	1.0	30°

Table 2.2 Single-Cage Example Load Flow Parameters

From this data, (2.8) shows that $I_{ds} = 1.0$ and (2.10) shows that $R_A = 1.0$. $\mathbf{D}$ can then be obtained from (2.15).

$$\mathbf{D} = \begin{bmatrix} 0.987 & 3.6 & 0 & 3.5 \\ -3.6 & -0.013 & -3.5 & 0 \\ 0 & 3.5 & -0.015 & 3.6 \\ -3.5 & 0 & -3.6 & -0.015 \end{bmatrix}$$

Solving (2.14) gives two non-zero solutions for ω_r. The zero eigenvalues can be neglected since they are not physically meaningful. The eigenvector x was scaled such that the I_{ds} entry is equal to its known value of 1.0. After scaling the eigenvector, the currents were rotated by δ. This was accomplished by calculating the vectors $I_k = (I_{dk} + jI_{qk})\,\mathrm{e}^{j\delta}$ for $k = \{s, r\}$. The rotated stator and rotor d-axis currents correspond to the real parts of I_k, and the rotated stator and rotor q-axis currents correspond to the imaginary parts of I_k. Finally, torque was calculated using (2.3).

The solutions for ω_r in radians per second and the currents and torque in per unit are presented below. Note that both the stable and unstable solutions are acquired rather than the single solution that would be obtained from an iterative method. These solutions were verified in the steady state equivalent circuit shown in Fig. 2.1 through the traditional method of iteration on the machine's slip until the proper terminal power was obtained.

Solution	$\omega_r \left(\frac{\text{rad}}{\text{sec}}\right)$	I_{ds}	I_{qs}	I_{dr}	I_{qr}	T_m
1	183.6	3.295	-3.708	-3.233	3.579	0.6801
2	370.7	1.110	0.0773	-0.999	-0.323	0.9839

Table 2.3 Solution to Single-Cage Induction Machine Example

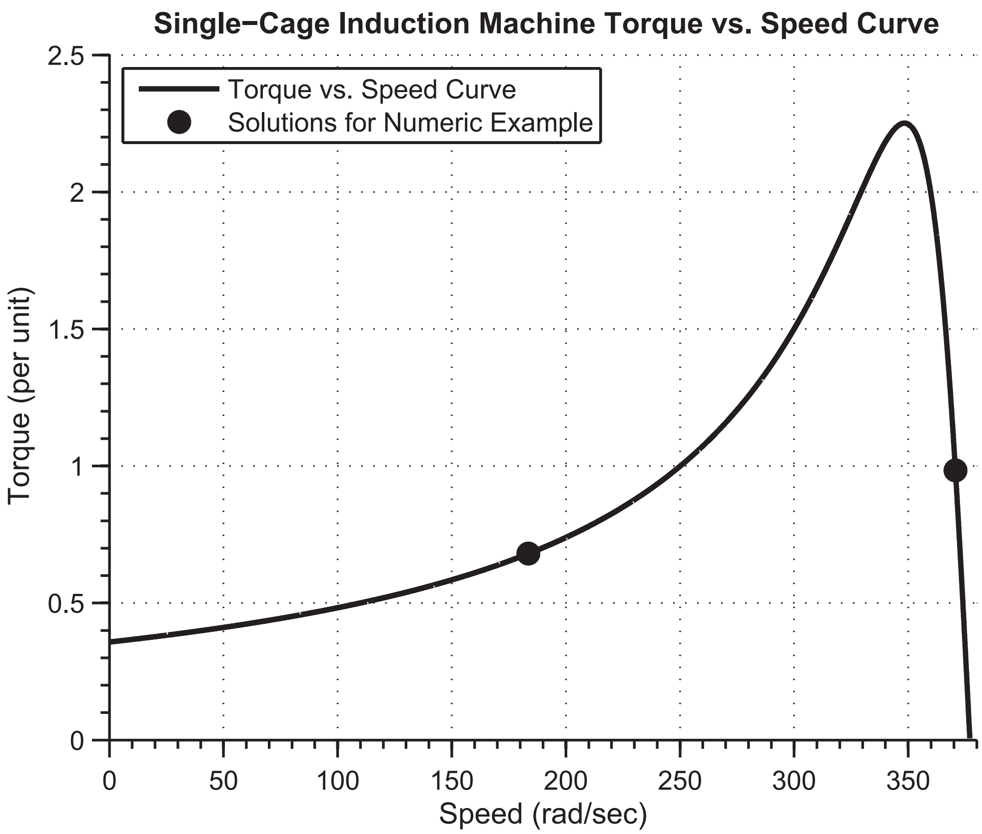

Figure 2.2 Torque Speed Curve and Solutions for Single-Cage Machine Numeric Example

The torque versus speed curve for this machine is given in Fig. 2.2, and both solutions are shown.

2.5 Dynamic Double-Cage Induction Machine Model

This section presents a dynamic double-cage induction machine model in the dq frame with linear magnetic relationships and short circuited rotor windings. The model was adapted from

[15]. This model is commonly used to represent the deep-bar effect and is similar to the model described by (2.1), (2.2), and (2.3) with an additional rotor circuit branch.

$$\begin{bmatrix} V_{ds} \\ V_{qs} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left(\begin{bmatrix} R_s & -(X_{ls}+X_m) & 0 & -X_m & 0 & -X_m \\ (X_{ls}+X_m) & R_s & X_m & 0 & X_m & 0 \\ 0 & -X_m & R_{r1} & -(X_{lr1}+X_m) & 0 & -X_m \\ X_m & 0 & (X_{lr1}+X_m) & R_{r1} & X_m & 0 \\ 0 & -X_m & 0 & -X_m & R_{r2} & -(X_{lr2}+X_m) \\ X_m & 0 & X_m & 0 & (X_{lr2}+X_m) & R_{r2} \end{bmatrix} \right.$$
$$\left. + \frac{\omega_r}{\omega_s} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_m & 0 & (X_{lr1}+X_m) & 0 & X_m \\ -X_m & 0 & -(X_{lr1}+X_m) & 0 & -X_m & 0 \\ 0 & X_m & 0 & X_m & 0 & (X_{lr2}+X_m) \\ -X_m & 0 & -X_m & 0 & -(X_{lr2}+X_m) & 0 \end{bmatrix} \right) \begin{bmatrix} I_{ds} \\ I_{qs} \\ I_{dr1} \\ I_{qr1} \\ I_{dr2} \\ I_{qr2} \end{bmatrix}$$
$$+ \frac{1}{\omega_s} \begin{bmatrix} (X_{ls}+X_m) & 0 & X_m & 0 & X_m & 0 \\ 0 & (X_{ls}+X_m) & 0 & X_m & 0 & X_m \\ X_m & 0 & (X_{lr1}+X_m) & 0 & X_m & 0 \\ 0 & X_m & 0 & (X_{lr1}+X_m) & 0 & X_m \\ X_m & 0 & X_m & 0 & (X_{lr2}+X_m) & 0 \\ 0 & X_m & 0 & X_m & 0 & (X_{lr2}+X_m) \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I_{ds} \\ I_{qs} \\ I_{dr1} \\ I_{qr1} \\ I_{dr2} \\ I_{qr2} \end{bmatrix} \tag{2.16}$$

$$\frac{2H}{\omega_s}\frac{\mathrm{d}\omega_\mathrm{r}}{\mathrm{d}t} = T_e\left(I_{ds}, I_{qs}, I_{dr1}, I_{qr1}, I_{dr2}, I_{qr2}\right) - T_m \tag{2.17}$$

$$T_e\left(I_{ds}, I_{qs}, I_{dr1}, I_{qr1}, I_{dr2}, I_{qr2}\right) = X_m\left(I_{qs}\left(I_{dr1}+I_{dr2}\right) - I_{ds}\left(I_{qr1}+I_{qr2}\right)\right) \tag{2.18}$$

The steady state equivalent circuit for the double-cage induction machine model given by (2.16), (2.17), and (2.18) is presented in Fig. 2.3.

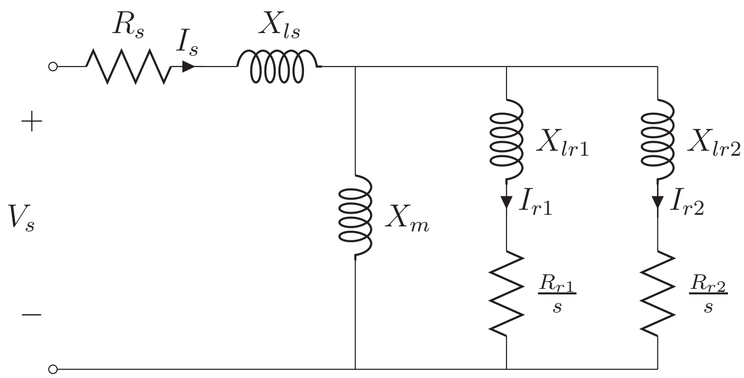

Figure 2.3 Double-Cage Induction Machine Steady State Equivalent Circuit

The dynamic double-cage induction machine model (2.16), (2.17), and (2.18) also fits into the general induction machine dynamic model framework given by (2.4) and (2.5). The development in section 2.3 can therefore be directly applied to the double-cage induction machine model.

2.6 Double-Cage Induction Machine Numeric Example

Assume V_{in} is obtained from a load flow analysis. Specify $V_{ds} = V_{in}$ and $V_{qs} = 0$. The matrix $\mathbf{D}$ is

$$\mathbf{D} = -\begin{bmatrix} (R_s - R_A) & -(X_{ls} + X_m) & 0 & -X_m & 0 & -X_m \\ (X_{ls} + X_m) & R_s & X_m & 0 & X_m & 0 \\ 0 & -X_m & R_{r1} & -(X_{lr1} + X_m) & 0 & -X_m \\ X_m & 0 & (X_{lr1} + X_m) & R_{r1} & X_m & 0 \\ 0 & -X_m & 0 & -X_m & R_{r2} & -(X_{lr2} + X_m) \\ X_m & 0 & X_m & 0 & (X_{lr2} + X_m) & R_{r2} \end{bmatrix} \tag{2.19}$$

Now ω_r and the currents in x can be found by solving (2.14). Note that the eigenvector x must be scaled correctly. Since the first entry of x should be I_{ds}, scaling can be done by multiplying the each entry in the eigenvector x by I_{ds} divided by the first entry of x.

Consider a double-cage induction machine with the following parameter values in per unit representation. These parameter values were obtained from the 90 kW double-cage machine in Table 2 of [15].

X_{ls}	X_{lr1}	X_{lr2}	X_m	R_s	R_{r1}	R_{r2}	ω_s (rad/sec)
0.0682	0.1206	0.0682	2.6595	0.0034	0.0130	0.1171	377

Table 2.4 Double-Cage Induction Machine Equivalent Circuit Parameters

Assume that the following per unit values are obtained from a load flow analysis.

P	V_{in}	δ
1.75	0.9	20°

Table 2.5 Double-Cage Example Load Flow Parameters

From this data, (2.8) shows that $I_{ds} = 1.9444$ and (2.10) shows that $R_A = 0.4629$. $\mathbf{D}$ can then be obtained from (2.19).

$$\mathbf{D} = \begin{bmatrix} 0.4595 & 2.7277 & 0 & 2.6595 & 0 & 2.6595 \\ -2.7277 & -0.0034 & -2.6595 & 0 & -2.6595 & 0 \\ 0 & 2.6595 & -0.013 & 2.7801 & 0 & 2.6595 \\ -2.6595 & 0 & -2.7801 & -0.013 & -2.6595 & 0 \\ 0 & 2.6595 & 0 & 2.6595 & -0.1171 & 2.7277 \\ -2.6595 & 0 & -2.6595 & 0 & -2.7277 & -0.1171 \end{bmatrix}$$

Solving (2.14) gives four non-zero solutions for ω_r. The zero eigenvalues can be neglected since they are not physically meaningful. The eigenvector x was scaled such that the I_{ds} entry is equal to its known value of 1.9444. After scaling the eigenvector, the currents were rotated by δ. This was accomplished by calculating the vectors $I_k = (I_{dk} + jI_{qk})\, \mathrm{e}^{j\delta}$ for $k = \{s, r1, r2\}$. The rotated d-axis currents correspond to the real parts of I_k, and the rotated q-axis currents correspond to the imaginary parts of I_k. Finally, torque was calculated using (2.18).

The solutions for ω_r in radians per second and the currents and torque in per unit are presented below. Note that four solutions (the high speed stable solution and three lower speed unstable solutions) are acquired rather than the single solution that would be obtained from an iterative method.

Solution	$\omega_r \left(\frac{\text{rad}}{\text{sec}}\right)$	I_{ds}	I_{qs}	I_{dr1}	I_{qr1}	I_{dr2}	I_{qr2}	T_m
1	-7.6	4.030	-5.388	-0.823	3.765	-3.188	1.448	1.5961
2	107.3	3.856	-4.908	-1.107	3.962	-2.725	0.759	1.6175
3	279.2	3.512	-3.964	-2.234	3.744	-1.248	0.0073	1.6547
4	364.8	2.237	-0.461	-1.954	0.197	-0.224	-0.039	1.7323

Table 2.6 Solution to Double-Cage Induction Machine Example

The torque versus speed curve for this machine is given in Fig. 2.4 with all four solutions shown.

2.7 Conclusion

A new method for determining the initial conditions of induction machine models in dynamic power system simulations has been developed. This method converts the initial conditions problem into a generalized eigenvalue formulation that can be solved using standard linear algebra techniques. In contrast to existing iterative methods, the eigenvalue method has the advantage of providing all solutions to the initial conditions problem and can reliability determine when no solutions exist. While eigenvalue solution techniques for arbitrarily sized matrices require iteration, these techniques are robust. Additionally, $\mathbf{D}^{-1}\mathbf{B}$ in (2.14) has rank two for the single-cage machine. Therefore, the largest factor in the eigenvalue characteristic equation for the single-cage induction machine model is a second order polynomial. Hence, the eigenvalue formulation can be non-iteratively solved using the quadratic formula. See Appendix A for the details of this solution technique. Similarly, the largest factor in the eigenvalue characteristic equation for the double-cage induction machine has a fourth order polynomial

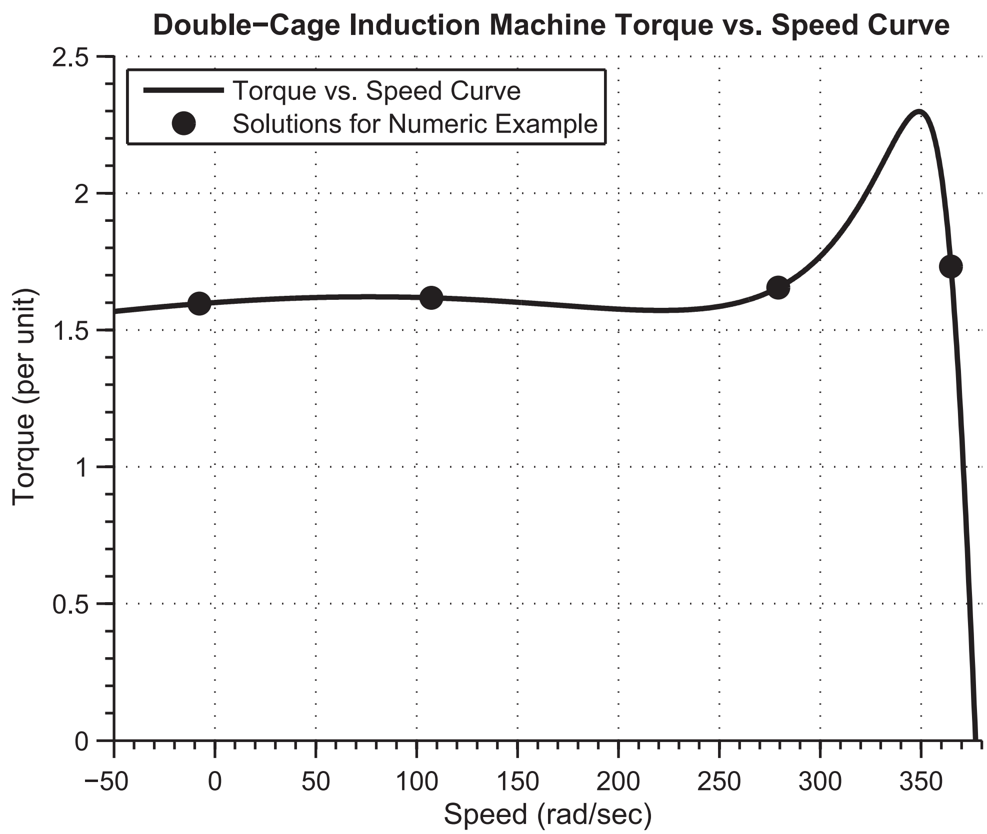

Figure 2.4 Torque Speed Curve and Solutions for Double-Cage Machine Numeric Example

which can be solved non-iteratively with the the equation for roots of quartic polynomials given in [20].

Chapter 3

A Multiparameter Eigenvalue Formulation of the Power Flow Equations

3.1 Development of the Power Flow Equations

The power flow equations relate the real and reactive power injected at each bus (with loads treated as negative injections) to the voltage magnitude and angle at each bus. The development of these equations is described in this section; see [9] for a textbook treatment of this material.

There are four variables associated with each bus k on a power system that are used in the power flow equations: the net real power injection (P_k), the net reactive power injection (Q_k), the voltage magnitude (V_k) and the voltage angle (δ_k).

Each bus k in the power system is classified as one of three bus types: slack bus, load (PQ) bus, and voltage controlled (PV) bus. A single bus is chosen as the slack bus. This bus has a fixed value of voltage magnitude V_k and angle δ_k. The real power P_k and reactive power Q_k injections for the slack bus are calculated from the power flow equations. For notational purposes, the slack bus is typically referred to as bus 1. The remaining buses are specified as either PQ or PV buses. A PQ bus has specified values for the real power injection P_k and Q_k. The voltage magnitude V_k and voltage angle δ_k are computed using the power flow equations. A PV bus has specified values of real power injection P_k and voltage magnitude V_k. The reactive power injection Q_k and voltage angle δ_k are computed using the power flow equations.

The electrical and topological properties of the transmission network are encapsulated in the admittance matrix $\mathbf{Y}$, where $\mathbf{Y}$ relates the voltages V and currents I in the network as

$V = \mathbf{Y}I$. The diagonal element of $\mathbf{Y}$ corresponding to bus k, Y_{kk}, is the sum of the admittances connected to bus k. The off-diagonal elements of $\mathbf{Y}$, Y_{kn} where $k \neq n$, are given by the negative of the sum of the admittances connected between buses k and n. The admittance matrix $\mathbf{Y}$ is typically decomposed into a real part $\mathbf{G} = \Re(\mathbf{Y})$ and imaginary part $\mathbf{B} = \Im(\mathbf{Y})$.

In most traditional derivations of the power flow equations, the voltage magnitude and angle are used directly. Here, however, the voltages at each bus are decomposed into orthogonal d and q components.

$$V_{dk} = V_k \cos(\delta_k) \tag{3.1}$$

$$V_{qk} = V_k \sin(\delta_k) \tag{3.2}$$

The power flow equations are then developed using the equation for complex power $P + jQ = VI^* = V\mathbf{Y}^*V^*$. For an n bus power system, bus i has equations,

$$P_i + jQ_i = (V_{di} + jV_{qi}) \sum_{k=1}^{n} (G_{ik} - jB_{ik})(V_{dk} - jV_{qk}) \tag{3.3}$$

Splitting (3.3) into real and imaginary parts and including the voltage magnitude relationship gives the full set of power flow equations.

$$P_i = V_{di} \sum_{k=1}^{n} (G_{ik}V_{dk} - B_{ik}V_{qk}) + V_{qi} \sum_{k=1}^{n} (B_{ik}V_{dk} + G_{ik}V_{qk}) \tag{3.4}$$

$$Q_i = V_{di} \sum_{k=1}^{n} (-B_{ik}V_{dk} - G_{ik}V_{qk}) + V_{qi} \sum_{k=1}^{n} (G_{ik}V_{dk} - B_{ik}V_{qk}) \tag{3.5}$$

$$V_i^2 = V_{di}^2 + V_{qi}^2 \tag{3.6}$$

With specified real power P_i and reactive power Q_i and no specification of voltage magnitude, PQ buses must satisfy (3.4) and (3.5). The voltage magnitude V_i is then calculated using (3.6). At PV buses, real power P_i and voltage magnitude V_i are specified, so (3.4) and (3.6) must be satisfied. The reactive power injection Q_i is then calculated from (3.5). The slack bus has specified V_{di} and V_{qi}, so the real power injection P_i, reactive power injection Q_i, and voltage magnitude V_i can be calculated using (3.4), (3.5), and (3.6).

3.2 A Multiparameter Eigenvalue Formulation of the Power Flow Equations

The power flow equations are now converted to a multiparameter eigenvalue form. Each bus contributes an additional two voltage parameters and an additional two equations that must be satisfied. Equations (3.7), (3.8) and (3.9) are the power flow equations presented in right multiparameter eigenvalue form for bus i and correspond to (3.4), (3.5), and (3.6), respectively. Equations (3.10), (3.11), and (3.12) are the power flow equations presented in left multiparameter eigenvalue form for bus i and correspond to (3.4), (3.5), and (3.6), respectively.

$$\begin{aligned}\Bigg(&\begin{bmatrix} 0 & 0 & -P_i \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{di}\begin{bmatrix} G_{ii} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + V_{qi}\begin{bmatrix} 0 & G_{ii} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &+ \sum_{k=1,\dots,n\ k\neq i} \left\{ V_{dk}\begin{bmatrix} G_{ik} & B_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + V_{qk}\begin{bmatrix} -B_{ik} & G_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}\Bigg) \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\end{aligned} \tag{3.7}$$

$$\begin{aligned}\Bigg(&\begin{bmatrix} 0 & 0 & -Q_i \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{di}\begin{bmatrix} -B_{ii} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + V_{qi}\begin{bmatrix} 0 & -B_{ii} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &+ \sum_{k=1,\dots,n\ k\neq i} \left\{ V_{dk}\begin{bmatrix} -B_{ik} & G_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + V_{qk}\begin{bmatrix} -G_{ik} & -B_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}\Bigg) \begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\end{aligned} \tag{3.8}$$

$$\left(\begin{bmatrix} 0 & 0 & -V_i^2 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{di}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + V_{qi}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} w_{1i} \\ w_{2i} \\ w_{3i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.9}$$

$$\begin{bmatrix} p_{1i} & p_{2i} & p_{3i} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & -P_i \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{di} \begin{bmatrix} G_{ii} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + V_{qi} \begin{bmatrix} 0 & G_{ii} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right.$$
$$\left. + \sum_{k=1,\ldots,n\ k\neq i} \left\{ V_{dk} \begin{bmatrix} G_{ik} & B_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + V_{qk} \begin{bmatrix} -B_{ik} & G_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \right) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \tag{3.10}$$

$$\begin{bmatrix} q_{1i} & q_{2i} & q_{3i} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & -Q_i \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{di} \begin{bmatrix} -B_{ii} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + V_{qi} \begin{bmatrix} 0 & -B_{ii} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right.$$
$$\left. + \sum_{k=1,\ldots,n\ k\neq i} \left\{ V_{dk} \begin{bmatrix} -B_{ik} & G_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + V_{qk} \begin{bmatrix} -G_{ik} & -B_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \right) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \tag{3.11}$$

$$\begin{bmatrix} r_{1i} & r_{2i} & r_{3i} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & -V_i^2 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{di} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + V_{qi} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \tag{3.12}$$

Both left and right formulations for bus i have eigenvalues V_{dk}, V_{qk} where $k = 2, \ldots, n$. Since the slack bus voltages V_{d1} and V_{q1} are known, the matrices not multiplied by an eigenvalue ($\mathbf{V_{i0}}$ in (1.7) and (1.8)) are given by (3.13) for the real power equations ((3.7) and (3.10)), and by (3.14) for the reactive power equations ((3.8) and (3.11)). The voltage equations ((3.9) and (3.12)) have easily identifiable constant matrices.

$$\begin{bmatrix} 0 & 0 & -P_i \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{d1} \begin{bmatrix} G_{i1} & B_{i1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + V_{q1} \begin{bmatrix} -B_{i1} & G_{i1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.13}$$

$$\begin{bmatrix} 0 & 0 & -Q_i \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{d1} \begin{bmatrix} -B_{i1} & G_{i1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + V_{q1} \begin{bmatrix} -G_{i1} & -B_{i1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.14}$$

The right formulation has eigenvectors $x = \begin{bmatrix} x_{1i} & x_{2i} & x_{3i} \end{bmatrix}^T$, $y = \begin{bmatrix} y_{1i} & y_{2i} & y_{3i} \end{bmatrix}^T$ and $w = \begin{bmatrix} w_{1i} & w_{2i} & w_{3i} \end{bmatrix}^T$ associated with the real power equation, reactive power equation, and voltage magnitude equation, respectively. The left formulation has eigenvectors $p = \begin{bmatrix} p_{1i} & p_{2i} & p_{3i} \end{bmatrix}^T$, $q = \begin{bmatrix} q_{1i} & q_{2i} & q_{3i} \end{bmatrix}^T$ and $r = \begin{bmatrix} r_{1i} & r_{2i} & r_{3i} \end{bmatrix}^T$ associated with the real power equation, reactive power equation, and voltage magnitude equation, respectively.

3.3 Left and Right Eigenvector Formulas

The eigenvalues of the multiparameter eigenvalue formulation can be easily recognized as the V_d and V_q voltages. However, the eigenvectors are not as easily identified. This section provides formulas for the eigenvectors x, y, w, p, q, and r in both left and right formulations.

Expressions for right eigenvectors can be obtained by expanding the second and third rows of the corresponding multiparameter eigenvalue equation and examining the relationships between the elements of the eigenvector. For instance, the second and third rows of the right multiparameter eigenvalue formulation of the real power equation (3.7) are

$$-V_{di}x_{2i} + V_{qi}x_{1i} = 0 \;\Rightarrow\; x_{2i} = \frac{V_{qi}}{V_{di}}x_{1i} \tag{3.15}$$

$$-x_{1i} + V_{di}x_{3i} = 0 \;\Rightarrow\; x_{3i} = \frac{1}{V_{di}}x_{1i} \tag{3.16}$$

Thus, the eigenvector x_i can be rewritten as

$$x_i = \begin{bmatrix} x_{1i} \\ \frac{V_{qi}}{V_{di}}x_{1i} \\ \frac{1}{V_{di}}x_{1i} \end{bmatrix} = x_{1i}\frac{1}{V_{di}} \begin{bmatrix} V_{di} \\ V_{qi} \\ 1 \end{bmatrix} \tag{3.17}$$

Since eigenvectors have a single degree of freedom in their magnitude (if v is an eigenvector, then av is also an eigenvector for scalar $a \neq 0$), assuming $V_{di} \neq 0$ and $x_{1i} \neq 0$, (3.17) can be rewritten as

$$x_i = \begin{bmatrix} V_{di} \\ V_{qi} \\ 1 \end{bmatrix} \tag{3.18}$$

Since the second and third rows of all right multiparameter formulations are identical, (3.7), (3.8), and (3.9) have identical eigenvectors

$$x_i = y_i = w_i = \begin{bmatrix} V_{di} \\ V_{qi} \\ 1 \end{bmatrix} \tag{3.19}$$

A similar process can be used for the left eigenvectors p, q, and r from (3.10), (3.11) and (3.12), respectively, where the second and third columns, rather than the second and third rows, are expanded. In contrast to the right eigenvectors, the left eigenvectors p_i, q_i, and r_i are not identical.

$$p_i = \begin{bmatrix} V_{di} \\ G_{ii}V_{qi} + \sum_{k=1\ k\neq i}^{n} \left(B_{ik}V_{dk} + G_{ik}V_{qk}\right) \\ P_i \end{bmatrix} \tag{3.20}$$

$$q_i = \begin{bmatrix} V_{di} \\ -B_{ii}V_{qi} + \sum_{k=1\ k\neq i}^{n} \left(G_{ik}V_{dk} - B_{ik}V_{qk}\right) \\ Q_i \end{bmatrix} \tag{3.21}$$

$$r_i = \begin{bmatrix} V_{di} \\ V_{qi} \\ V_i^2 \end{bmatrix} \tag{3.22}$$

One advantage of calculating the eigenvectors using (3.19) through (3.22) instead of using a numeric routine like MATLAB's `EIG` function is that eigenvector formulas represent the exact eigenvector values rather than the approximation obtained from a numeric eigenvector solver.

An additional point to note is that the eigenvector formulas indicate that the eigenvectors must be real.

3.4 Direct Solution of the Multiparameter Eigenvalue Formulation for Two Bus Systems

Since the slack bus voltages are known, the power flow equations for a two bus systems can be represented with a two-parameter eigenvalue problem (V_{d2} and V_{q2} are the eigenvalues). As described in Section 1.1.3, two parameter problems can be decoupled into a system of two generalized eigenvalue problems using the Kronecker product method. Since each matrix in the generalized eigenvalue form has dimension 3×3, the Kronecker product method yields generalized eigenvalue problems which have matrices of dimension 9×9. Generalized eigenvalue problems of this size are easily solvable. Joint solutions of the two generalized eigenvalue problems are solutions of the two bus system power flow equations. This section explores the Kronecker product method for two bus systems and provides numeric examples. Systems with both PQ and PV buses are investigated.

A two bus power system has a 2×2 admittance matrix $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$. Bus 1 is a slack bus with known values of V_{d1} and V_{q1}. First consider bus 2 as a PQ bus with real power injection P_2 and reactive power injection Q_2. Solving the power flow equations requires finding the voltages at bus 2: V_{d2} and V_{q2}. Using the P and Q multiparameter eigenvalue formulations given in (3.7) and (3.8) as well as the Kronecker product method in (1.26) and (1.27), the following generalized eigenvalue problems must be satisfied

$$\mathbf{\Delta_1} z = V_{d2} \mathbf{\Delta_0} z \tag{3.23}$$

$$\mathbf{\Delta_2} z = V_{q2} \mathbf{\Delta_0} z \tag{3.24}$$

where

$$\mathbf{\Delta_0} = \begin{bmatrix} 0 & -G_{22}B_{22} & 0 & G_{22}B_{22} & 0 & 0 & 0 & 0 & 0 \\ G_{22} & 0 & 0 & 0 & G_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -G_{22} & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.25}$$

$$\mathbf{\Delta_1} = \begin{bmatrix} 0 & aB_{22} & 0 & -bG_{22} & (aG_{22}+bB_{22}) & -Q_2G_{22} & 0 & -P_2B_{22} & 0 \\ -a & 0 & 0 & -b & 0 & 0 & P_2 & 0 & 0 \\ 0 & 0 & 0 & -G_{22} & 0 & 0 & 0 & 0 & 0 \\ -b & a & -Q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -B_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.26}$$

$$\mathbf{\Delta_2} = \begin{bmatrix} (-aB_{22}+bG_{22}) & -aG_{22} & Q_2G_{22} & -bB_{22} & 0 & 0 & P_2B_{22} & 0 & 0 \\ 0 & -a & 0 & 0 & -b & 0 & 0 & P_2 & 0 \\ G_{22} & 0 & a & 0 & 0 & b & 0 & 0 & -P_2 \\ 0 & 0 & 0 & -b & a & -Q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 & 0 & 0 & b & -a & Q_2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{3.27}$$

and, for notational convenience, $a = G_{21}V_{d1} - B_{21}V_{q1}$ and $b = B_{21}V_{d1} + G_{21}V_{q1}$.

To determine which pairs of eigenvalues are actually solutions to the power flow equations, test each of the possible combinations for the voltages (V_{d2},V_{q2}) in the power flow equations (3.4) and (3.5). Combinations of voltages that yield the specified values of real and reactive power injections P_i and Q_i are solutions to the power flow equations.

Next, a numeric example of the Kronecker product method with a PQ bus is presented. Consider a two bus system where the buses are connected by a transmission line with impedance $0.01 + j0.05$ per unit. This gives the admittance matrix

$$\mathbf{G} + j\mathbf{B} = \begin{bmatrix} 3.846 & -3.846 \\ -3.846 & 3.846 \end{bmatrix} + j \begin{bmatrix} -19.231 & 19.231 \\ 19.231 & -19.231 \end{bmatrix} \tag{3.28}$$

Assume a slack bus voltage $V_{d1} + jV_{q1} = 1.0 + j0$ per unit and bus 2 power injection of $P_2 + jQ_2 = -0.75 - j0.5$ per unit (negative indicating bus 2 has a load). Then $\mathbf{\Delta_0}$, $\mathbf{\Delta_1}$, and $\mathbf{\Delta_2}$ can be determined from (3.25), (3.26), and (3.27).

$$\mathbf{\Delta_0} = \begin{bmatrix} 0 & 73.96 & 0 & -73.96 & 0 & 0 & 0 & 0 & 0 \\ 3.846 & 0 & 0 & 0 & 3.846 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.846 & 0 & 0 & 0 \\ -19.23 & 0 & 0 & 0 & -19.23 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 19.23 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.29}$$

$$\mathbf{\Delta_1} = \begin{bmatrix} 0 & 73.96 & 0 & -73.96 & -384.6 & 1.923 & 0 & -14.42 & 0 \\ 3.846 & 0 & 0 & -19.23 & 0 & 0 & -0.75 & 0 & 0 \\ 0 & 0 & 0 & -3.846 & 0 & 0 & 0 & 0 & 0 \\ -19.23 & -3.846 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 19.23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.30}$$

$$\mathbf{\Delta_2} = \begin{bmatrix} 0 & 14.79 & -1.923 & 369.8 & 0 & 0 & 14.42 & 0 & 0 \\ 0 & 3.846 & 0 & 0 & -19.23 & 0 & 0 & -0.75 & 0 \\ 3.846 & 0 & -3.846 & 0 & 0 & 19.23 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & -19.23 & -3.846 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -19.23 & 0 & 0 & 0 & 0 & 0 & 19.23 & 3.846 & -0.5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{3.31}$$

Solving (3.23) and (3.24) gives the sets of possible values for V_{d2} and V_{q2}. V_{d2} must be in the set $\{0,\ 0.0348,\ 0.9652\}$ and V_{q2} must be in the set $\{-5,\ -0.0325,\ 0.2,\ 11.5573\}$. By verifying whether the calculated real and reactive power injections from each possible combination from these two sets matches the specified power injections, the actual solutions of $V_{d2} + jV_{q2} = 0.9652 - j0.0325$ and $V_{d2} + jV_{q2} = 0.0348 - j0.0325$ are found.

A similar process can be done for a two bus system where bus 2 is a PV bus with real power injection P_2 and voltage magnitude V_2. The Kronecker product equations (3.23) and (3.23) must be solved, where

$$\mathbf{\Delta_0} = \begin{bmatrix} 0 & G_{22} & 0 & -G_{22} & 0 & 0 & 0 & 0 & 0 \\ G_{22} & 0 & 0 & 0 & G_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -G_{22} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.32}$$

$$\mathbf{\Delta_1} = \begin{bmatrix} 0 & -a & 0 & 0 & -b & -V_2^2 G_{22} & 0 & P_2 & 0 \\ -a & 0 & 0 & -b & 0 & 0 & P_2 & 0 & 0 \\ 0 & 0 & 0 & -G_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -V_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.33}$$

$$\mathbf{\Delta_2} = \begin{bmatrix} a & 0 & V_2^2 G_{22} & b & 0 & 0 & -P_2 & 0 & 0 \\ 0 & -a & 0 & 0 & -b & 0 & 0 & P_2 & 0 \\ G_{22} & 0 & a & 0 & 0 & b & 0 & 0 & -P_2 \\ 0 & 0 & 0 & 0 & 0 & -V_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_2^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{3.34}$$

and, for notational convenience, $a = G_{21}V_{d1} - B_{21}V_{q1}$ and $b = B_{21}V_{d1} + G_{21}V_{q1}$.

Then each combination of eigenvalues must be checked in the power flow equations for real power (3.4) and voltage magnitude (3.6). Combinations that satisfy these equations are the solutions to the power flow equations.

Next, a numeric example of the Kronecker product method for a two bys system with a PV bus is presented. Consider the same system as described in the PQ bus case. Assume a bus 2 power injection of $P_2 = 3.0$ per unit (positive indicating bus 2 has a generator) and bus 2 voltage magnitude $V_2 = 0.9$ per unit. Then $\mathbf{\Delta_0}$, $\mathbf{\Delta_1}$, and $\mathbf{\Delta_2}$ can be determined from (3.32), (3.33), and (3.34).

$$\mathbf{\Delta_0} = \begin{bmatrix} 0 & 3.846 & 0 & -3.846 & 0 & 0 & 0 & 0 & 0 \\ 3.846 & 0 & 0 & 0 & 3.846 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.846 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.35}$$

$$\mathbf{\Delta_1} = \begin{bmatrix} 0 & 3.846 & 0 & 0 & -19.23 & -3.115 & 0 & 3.0 & 0 \\ 3.846 & 0 & 0 & -19.23 & 0 & 0 & 3.0 & 0 & 0 \\ 0 & 0 & 0 & -3.846 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.81 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.36}$$

$$\mathbf{\Delta_2} = \begin{bmatrix} -3.846 & 0 & 3.115 & 19.23 & 0 & 0 & -3.0 & 0 & 0 \\ 0 & 3.846 & 0 & 0 & -19.23 & 0 & 0 & 3.0 & 0 \\ 3.846 & 0 & -3.846 & 0 & 0 & 19.23 & 0 & 0 & -3.0 \\ 0 & 0 & 0 & 0 & 0 & -0.81 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.81 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{3.37}$$

Solving (3.23) and (3.24) gives the sets of possible values for V_{d2} and V_{q2}. V_{d2} must be in the set $\{-0.8813,\ 0,\ 0.8837\}$ and V_{q2} must be in the set $\{-0.1823,\ 0.0867,\ 0.1707,\ 1.6822,\ 7.9643\}$. By verifying whether the calculated real and reactive power injections from each possible combination from these two sets matches the specified real power injection and voltage magnitude, the actual solutions of $V_{d2} + jV_{q2} = 0.8837 + j0.1707$ and $V_{d2} + jV_{q2} = -0.8813 - j0.1823$ are found.

3.5 A Reformulation of the Power Flow Equations Motivated by Eigenvalue Problem Structure

The right multiparameter eigenvalue formulation of the power flow equations (3.7), (3.8), (3.9) can be reformulated to resemble a standard eigenvalue problem with a matrix that is dependent on the V_d and V_q voltages. Although this reformulation does not yet have any practical applications, it is an interesting theoretical development. Motivated by standard eigenvalue problem structure, this section develops a method for reformulating the power flow equations and provides an example for a three bus system.

Start with the formulation of the power flow equations for bus i shown in (3.38). This formulation takes advantage of the fact that the right eigenvectors are the same for the real power equation, reactive power equation, and voltage magnitude equation. Two relationships (real power and reactive power for PQ buses, real power and voltage magnitude for PV buses) are combined into one block matrix equation, and the known expressions are substituted for the eigenvalues as discussed in Section 3.3.

$$\begin{bmatrix} \mathbf{A_1} & \cdots & \mathbf{A_i} & \cdots & \mathbf{A_n} \end{bmatrix} \overline{\mathbf{V}} \begin{bmatrix} V_{di} \\ V_{qi} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.38}$$

where n is the number of buses in the system, $\overline{\mathbf{V}}$ is a $3n \times 3$ matrix

$$\overline{V} = \begin{bmatrix} V_{d1}\mathbf{I_3} \\ V_{q1}\mathbf{I_3} \\ 1\mathbf{I_3} \\ \vdots \\ V_{di}\mathbf{I_3} \\ V_{qi}\mathbf{I_3} \\ 1\mathbf{I_3} \\ \vdots \\ V_{dn}\mathbf{I_3} \\ V_{qn}\mathbf{I_3} \\ 1\mathbf{I_3} \end{bmatrix} \tag{3.39}$$

and, if bus i is a PQ bus,

$$\mathbf{A_k} = \begin{bmatrix} G_{ik} & B_{ik} & 0 & -B_{ik} & G_{ik} & 0 & 0 & 0 & 0 \\ -B_{ik} & G_{ik} & 0 & -G_{ik} & -B_{ik} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad k = 1, \ldots, n, \ k \neq i \tag{3.40}$$

$$\mathbf{A_i} = \begin{bmatrix} G_{ii} & 0 & 0 & 0 & G_{ii} & 0 & 0 & 0 & -P_i \\ -B_{ii} & 0 & 0 & 0 & -B_{ii} & 0 & 0 & 0 & -Q_i \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \tag{3.41}$$

or, if bus i is a PV bus

$$\mathbf{A_k} = \begin{bmatrix} G_{ik} & B_{ik} & 0 & -B_{ik} & G_{ik} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad k = 1, \ldots, n, \ k \neq i \tag{3.42}$$

$$\mathbf{A_i} = \begin{bmatrix} G_{ii} & 0 & 0 & 0 & G_{ii} & 0 & 0 & 0 & -P_i \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -V_i^2 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \tag{3.43}$$

(3.38) is a nullspace equation. It shows that the vector $\overline{\mathbf{V}} \begin{bmatrix} V_{di} \\ V_{qi} \\ 1 \end{bmatrix}$ is in the nullspace of the matrix $\begin{bmatrix} \mathbf{A_1} & \cdots & \mathbf{A_i} & \cdots & \mathbf{A_n} \end{bmatrix}$. Since this matrix is completely composed of known values, the nullspace can be calculated immediately and written as

$$\overline{\mathbf{V}}\begin{bmatrix}V_{di}\\V_{qi}\\1\end{bmatrix} = \alpha_{i1}z_{i1} + \alpha_{i2}z_{i2} + \cdots + \alpha_{ik}z_{ik} \tag{3.44}$$

where $k = \dim\left(\ker\left(\begin{bmatrix}\mathbf{A_1} & \cdots & \mathbf{A_i} & \cdots & \mathbf{A_n}\end{bmatrix}\right)\right)$, α_i are scalars, and z_i refer to a set of vectors forming a basis of the nullspace and composed entirely of the known parameters G, B, P_i, Q_i, and V_i.

The values for the scalars α_i can be found solely in terms of the V_d and V_q voltages by expanding the vector $\overline{\mathbf{V}}\begin{bmatrix}V_{di}\\V_{qi}\\1\end{bmatrix}$ and comparing entries to the nullspace described in (3.44).

Next, the psuedo-inverse can be used to isolate $\begin{bmatrix}V_{di}\\V_{qi}\\1\end{bmatrix}$ in (3.44). This entails multiplication of both sides of (3.44) by $\overline{\mathbf{V}}^T$.

$$\overline{\mathbf{V}}^T\overline{\mathbf{V}}\begin{bmatrix}V_{di}\\V_{qi}\\1\end{bmatrix} = \overline{\mathbf{V}}^T\left(\alpha_{i1}z_{i1} + \alpha_{i2}z_{i2} + \cdots + \alpha_{ik}z_{ik}\right) \tag{3.45}$$

Hence,

$$\begin{bmatrix}V_{di}\\V_{qi}\\1\end{bmatrix} = \left(\overline{\mathbf{V}}^T\overline{\mathbf{V}}\right)^{-1}\overline{\mathbf{V}}^T\left(\alpha_{i1}z_{i1} + \alpha_{i2}z_{i2} + \cdots + \alpha_{ik}z_{ik}\right) \tag{3.46}$$

Expansion of $\left(\overline{\mathbf{V}}^T\overline{\mathbf{V}}\right)^{-1}$ gives

$$\left(\overline{\mathbf{V}}^T\overline{\mathbf{V}}\right)^{-1} = \frac{1}{d}\mathbf{I_3} \tag{3.47}$$

where $d = V_{d1}^2 + V_{q1}^2 + 1 + V_{d2}^2 + V_{q2}^2 + 1 + \ldots + V_{dn}^2 + V_{qn}^2 + 1$.

Therefore, (3.46) can be rewritten as

$$\begin{bmatrix} V_{di} \\ V_{qi} \\ 1 \end{bmatrix} = \frac{1}{d}\overline{\mathbf{V}}^T \left(\alpha_{i1} z_{i1} + \alpha_{i2} z_{i2} + \cdots + \alpha_{ik} z_{ik}\right) \tag{3.48}$$

Collecting terms on the right-hand side of (3.48) gives

$$\begin{bmatrix} V_{di} \\ V_{qi} \\ 1 \end{bmatrix} = \frac{1}{d}\left[\mathbf{f_i}\left(\alpha_i, G, B, P_i, Q_i\right)\right] \begin{bmatrix} V_{d2} \\ V_{q2} \\ 1 \\ V_{d3} \\ V_{q3} \\ 1 \\ \vdots \\ V_{dn} \\ V_{qn} \\ 1 \end{bmatrix} \tag{3.49}$$

where $\left[\mathbf{f_i}\left(\alpha_i, G, B, P_i, Q_i\right)\right]$ is a $3 \times 3(n-1)$ matrix. Note that the slack bus voltages V_{d1} and V_{q1} do not need to be separated out since they are known values and can therefore be grouped with the terms multiplying a constant 1 in (3.49). All of the constants in (3.49) could be consolidated into one term; they are kept separate to maintain the form of stacked eigenvectors.

Repeating this process for all buses and combining the corresponding equations from (3.49) gives the power flow equations in a form that resembles a standard eigenvalue problem as shown in (3.50). d is similar to an eigenvalue, and the vectors containing the V_d and V_q voltages resemble eigenvectors. All α_i are known functions of V_d and V_q and are therefore eliminated by substitution.

$$d\begin{bmatrix} V_{d2} \\ V_{q2} \\ 1 \\ V_{d3} \\ V_{q3} \\ 1 \\ \vdots \\ V_{dn} \\ V_{qn} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{f_2}\left(V_d, V_q, G, B, P, Q\right) \\ \mathbf{f_3}\left(V_d, V_q, G, B, P, Q\right) \\ \vdots \\ \mathbf{f_n}\left(V_d, V_q, G, B, P, Q\right) \end{bmatrix} \begin{bmatrix} V_{d2} \\ V_{q2} \\ 1 \\ V_{d3} \\ V_{q3} \\ 1 \\ \vdots \\ V_{dn} \\ V_{qn} \\ 1 \end{bmatrix} \tag{3.50}$$

Next, an example of (3.50) is presented for a three bus system where bus 1 is the slack bus and buses 2 and 3 are PQ buses. The reformulation is given by

$$d\begin{bmatrix} V_{d2} \\ V_{q2} \\ 1 \\ V_{d3} \\ V_{q3} \\ 1 \end{bmatrix} = \begin{bmatrix} V_{d2}^2 & V_{d2}V_{q2} & a_1 & V_{d3}V_{d2} & V_{q3}V_{d2} & 0 \\ V_{d2}V_{q2} & V_{q2}^2 & a_2 & V_{d3}V_{q2} & V_{q3}V_{q2} & 0 \\ V_{d2} & V_{q2} & a_3 & V_{d3} & V_{q3} & 0 \\ V_{d2}V_{d3} & V_{q2}V_{d3} & 0 & V_{d3}^2 & V_{d3}V_{q3} & b_1 \\ V_{d2}V_{q3} & V_{q2}V_{q3} & 0 & V_{d3}V_{q3} & V_{q3}^2 & b_2 \\ V_{d2} & V_{q2} & 0 & V_{d3} & V_{q3} & b_3 \end{bmatrix} \begin{bmatrix} V_{d2} \\ V_{q2} \\ 1 \\ V_{d3} \\ V_{q3} \\ 1 \end{bmatrix} \tag{3.51}$$

where

$$d = V_{d1}^2 + V_{q1}^2 + 1 + V_{d2}^2 + V_{q2}^2 + 1 + V_{d3}^2 + V_{q3}^2 + 1$$

$$\begin{aligned} a_1 = 3V_{d2} + V_{q1}^2 V_{d2} + V_{d1}\Bigg(&-V_{q1}V_{q2} - \frac{B_{21}B_{23} + G_{21}G_{23}}{G_{21}^2 + B_{21}^2}\left(V_{d2}V_{d3} + V_{q2}V_{q3}\right) \\ &+\frac{B_{21}G_{23} - G_{21}B_{23}}{G_{21}^2 + B_{21}^2}\left(V_{q2}V_{d3} - V_{d2}V_{q3}\right) - \frac{G_{22}G_{21} + B_{22}B_{21}}{G_{21}^2 + B_{21}^2}\left(V_{d2}^2 + V_{q2}^2\right) + \frac{P_2G_{21} - B_{21}Q_{21}}{G_{21}^2 + B_{21}^2}\Bigg) \end{aligned}$$

$$a_2 = 3V_{q2} + V_{q1}^2 V_{q2} + V_{d1} \left(V_{q1}V_{d2} + \frac{G_{21}B_{23} - B_{21}G_{23}}{G_{21}^2 + B_{21}^2} \left(V_{d3}V_{d2} + V_{q3}V_{q2}\right) \right.$$
$$\left. - \frac{B_{21}B_{23} + G_{21}G_{23}}{G_{21}^2 + B_{21}^2} \left(V_{q2}V_{d3} - V_{q3}V_{d2}\right) + \frac{B_{22}G_{21} - G_{22}B_{21}}{G_{21}^2 + B_{21}^2} \left(V_{d2}^2 + V_{q2}^2\right) + \frac{P_2B_{21} + Q_2G_{21}}{G_{21}^2 + B_{21}^2} \right)$$

$$a_3 = 3 + V_{d1}^2 + V_{q1}^2$$

$$b_1 = 3V_{d3} + V_{q1}^2 V_{d3} + V_{d1} \left(-V_{q1}V_{q3} - \frac{B_{31}B_{32} + G_{31}G_{32}}{G_{31}^2 + B_{31}^2} \left(V_{d2}V_{d3} + V_{q2}V_{q3}\right) \right.$$
$$\left. + \frac{B_{31}G_{32} - G_{31}B_{32}}{G_{31}^2 + B_{31}^2} \left(V_{d2}V_{q3} - V_{q2}V_{d3}\right) - \frac{B_{31}B_{33} + G_{31}G_{33}}{G_{31}^2 + B_{31}^2} \left(V_{d3}^2 + V_{q3}^2\right) + \frac{P_3G_{31} - Q_3B_{31}}{G_{31}^2 + B_{31}^2} \right)$$

$$b_2 = 3V_{q3} + V_{q1}^2 V_{q3} + V_{d1} \left(V_{q1}V_{d3} + \frac{G_{31}B_{32} - B_{31}G_{32}}{G_{31}^2 + B_{31}^2} \left(V_{d2}V_{d3} + V_{q2}V_{q3}\right) \right.$$
$$\left. + \frac{B_{31}B_{32} + G_{31}G_{32}}{G_{31}^2 + B_{31}^2} \left(V_{q2}V_{d3} - V_{d2}V_{q3}\right) + \frac{G_{31}B_{33} - B_{31}G_{33}}{G_{31}^2 + B_{31}^2} \left(V_{d3}^2 + V_{q3}^2\right) + \frac{P_3B_{31} + Q_3G_{31}}{G_{31}^2 + B_{31}^2} \right)$$

$$b_3 = 3 + V_{q1}^2 + V_{d1}^2$$

The hypothesis that the formulation of the power flow equations given by (3.50) could lead to a new iterative approach to solving the power flow equations was investigated. Starting from initial values of V_d and V_q that were close to a solution, the solution might be found by evaluating the eigenvalue problem, substituting the new V_d and V_q values from the solution of the eigenvalue problem back into (3.50), reevaluating the eigenvalue problem, and repeating. If this process converged, it may have provided a new method for solving the power flow equations. Unfortunately, all investigated test cases diverged with V_d and V_q values going to positive or negative infinity. Therefore, this reformulation of the power flow equations does not appear to enable an iterative solution method. Future work includes further investigation into this reformulation of the power flow equations to determine whether it has any practical applications.

3.6 Power Flow Equations Eigenvalue Sensitivity

Since the eigenvalues of the multiparameter eigenvalue form of the power flow equations are the V_d and V_q voltages, an eigenvalue sensitivity analysis yields a linearization of the power flow equations. This linearization gives a first order approximation to how the V_d and V_q voltages change with a change in real power ΔP, reactive power ΔQ, and slack bus voltages ΔV_{d1} and ΔV_{q1}. The eigenvalue sensitivity linearization provides identical results to the linearization obtained from the power flow Jacobian. This section contains an eigenvalue sensitivity analysis and details its equivalence to the Jacobian linearization.

To develop the eigenvalue sensitivity linearization, start from the eigenvalue sensitivity equation for a general multiparameter eigenvalue problem given in (1.32). Applying (1.32) for the right multiparameter eigenvalue form of the power flow equations (3.7), (3.8), (3.9) requires the expressions for the left and right eigenvectors from (3.19), (3.20), (3.21), and (3.22) as well as the expressions for the constant matrices (3.13) and (3.14).

The eigenvalue sensitivity linearization for the multiparameter eigenvalue formulation of the power flow equations for an n bus system has the form

$$\mathbf{A_{sens}} \Delta V_{sens} = b_{sens} \tag{3.52}$$

where $\mathbf{A_{sens}}$ is a $2\,(n-1) \times 2\,(n-1)$ matrix, b_{sens} is a vector with length $2\,(n-1)$, and the vector ΔV_{sens} is

$$\Delta V_{sens} = \begin{bmatrix} \Delta V_{d2} \\ \Delta V_{q2} \\ \vdots \\ \Delta V_{dn} \\ \Delta V_{qn} \end{bmatrix}$$

Each bus has two associated equations, and thus two corresponding rows in $\mathbf{A_{sens}}$ and b_{sens}. If the i^{th} row of (3.52) refers to a real power equation, it is given by

$$
\begin{aligned}
&\left[\quad p_i^T \begin{bmatrix} G_{i2} & B_{i2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_i \quad p_i^T \begin{bmatrix} -B_{i2} & G_{i2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_i \quad \cdots \right. \\
&\cdots \quad p_i^T \begin{bmatrix} G_{ii} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_i \quad p_i^T \begin{bmatrix} 0 & G_{ii} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_i \quad \cdots \\
&\left. \cdots \quad p_i^T \begin{bmatrix} G_{in} & B_{in} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_i \quad p_i^T \begin{bmatrix} -B_{in} & G_{in} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_i \quad \right] \Delta V_{sens} \\
&= p_i^T \left(\begin{bmatrix} 0 & 0 & -\Delta P_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Delta V_{d1} \begin{bmatrix} -B_{i1} & G_{i1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Delta V_{q1} \begin{bmatrix} -B_{i1} & G_{i1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) x_i
\end{aligned}
\tag{3.53}
$$

or, if the i^{th} row refers to a reactive power equation,

$$\begin{bmatrix} q_i^T \begin{bmatrix} -B_{i2} & G_{i2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} y_i & q_i^T \begin{bmatrix} -G_{i2} & -B_{i2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} y_i & \cdots \\ \cdots & q_i^T \begin{bmatrix} -B_{ii} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} y_i & q_i^T \begin{bmatrix} 0 & -B_{ii} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} y_i \quad \cdots \\ \cdots & q_i^T \begin{bmatrix} -B_{in} & G_{in} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} y_i & q_i^T \begin{bmatrix} -G_{in} & -B_{in} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} y_i \end{bmatrix} \Delta V_{sens} \tag{3.54}$$

$$= q_i^T \left(\begin{bmatrix} 0 & 0 & -\Delta Q_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Delta V_{d1} \begin{bmatrix} -B_{i1} & G_{i1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Delta V_{q1} \begin{bmatrix} -G_{i1} & -B_{i1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) y_i$$

or, if the i^{th} row refers to a voltage magnitude equation

$$\begin{bmatrix} 0 & 0 & \cdots & r_i^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} w_i & r_i^T \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} w_i & \cdots & 0 & 0 \end{bmatrix} \Delta V_{sens}$$

$$= r_i^T \begin{bmatrix} 0 & 0 & -\Delta V_i^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} w_i \tag{3.55}$$

To show that the eigenvalue sensitivity linearization is equivalent to the linearization obtained from the Jacobian, substitute (3.19) for the eigenvectors x, y, w and (3.20), (3.21), and (3.22) for p, q, r, respectively, and expand the resulting expressions. Then, the i^{th} row is equivalent to

$$V_{di}\begin{bmatrix}\frac{\partial P_i}{\partial V_{d2}} & \frac{\partial P_i}{\partial V_{q2}} & \cdots & \frac{\partial P_i}{\partial V_{dn}} & \frac{\partial P_i}{\partial V_{qn}}\end{bmatrix}\Delta V_{sens} = V_{di}\Delta P_i \tag{3.56}$$

if the i^{th} row refers to a real power equation,

$$V_{di}\begin{bmatrix}\frac{\partial Q_i}{\partial V_{d2}} & \frac{\partial Q_i}{\partial V_{q2}} & \cdots & \frac{\partial Q_i}{\partial V_{dn}} & \frac{\partial Q_i}{\partial V_{qn}}\end{bmatrix}\Delta V_{sens} = V_{di}\Delta Q_i \tag{3.57}$$

if the i^{th} row refers to a reactive power equation, or

$$V_{di}\begin{bmatrix}0 & 0 & \cdots & \frac{\partial V_i}{\partial V_{di}} & \frac{\partial V_i}{\partial V_{qi}} & \cdots & 0 & 0\end{bmatrix}\Delta V_{sens} = V_{di}\Delta V_i^2 \tag{3.58}$$

if the i^{th} row refers to a voltage magnitude equation.

Equations (3.56), (3.57), and (3.58) are those of the standard Jacobian linearization of the power flow equations with both sides multiplied by V_{di}. Therefore, the eigenvalue sensitivity linearization gives identical results to the standard Jacobian linearization.

3.7 Conclusions and Future Work

This chapter first detailed a new multiparameter eigenvalue formulation of the power flow equations for both PQ and PV buses. Formulas for both the left and right eigenvectors have been developed.

Next, the Kronecker product method for direct solution of the power flow equations for two bus systems with both PQ and PV buses has been explored. This method converts the two parameter eigenvalue problem into two generalized eigenvalue problems that must be simultaneously satisfied. Since the matrices in the multiparameter eigenvalue formulation have size 3×3, the generalized eigenvalue problems have 9×9 matrices that can be solved using existing generalized eigenvalue solution techniques. Since no solution techniques for multiparameter eigenvalue problems with more than two parameters were found, there is currently no known method for direct solutions of the multiparameter eigenvalue formulation for systems with more than two buses.

Motivated by eigenvalue problem structure, the expressions for the right eigenvectors were next used to convert the multiparameter eigenvalue problem to a form that resembled a standard eigenvalue problem with matrix elements that are functions of the V_d and V_q voltages. Although this reformulation does not have any known practical applications, it is an interesting theoretical development.

Finally, an eigenvalue sensitivity analysis of the multiparameter eigenvalue formulation was performed. The linearization resulting from this sensitivity analysis gave identical results to the linearization obtained from the Jacobian method.

An eigenvector sensitivity analysis was also attempted. In addition to containing the derivation for the eigenvalue sensitivity for multiparameter eigenvalue problems, [11] also described the right eigenvector sensitivities. Since the real power equation right eigenvector x_i is identical to the reactive power equation right eigenvector y_i (as described in Section 3.3, both are $\begin{bmatrix} V_{di} & V_{qi} & 1 \end{bmatrix}^T$), the eigenvector sensitivity analysis was expected to show that the eigenvector sensitivities for x_i and y_i were also identical. Furthermore, since the third entries of x_i and y_i are constant values of one (when properly scaled), the eigenvector sensitivities were expected to show that the the sensitivity of this entry was zero. However, neither of these characteristics were found in an eigenvector sensitivity analysis of a three bus system with a slack bus and two PQ buses. Additionally, since the eigenvector entries are known to be the voltage components V_{di} and V_{qi}, a presumed sensitivity of the eigenvectors can be obtained from the Jacobian linearization of the power flow equations. The sensitivity results obtained in this way did not satisfy the eigenvector sensitivity equations described in [11]. It is possible that the multiparameter eigenvalue formulation of the power flow equations does not meet some criteria required by the eigenvector sensitivity analysis, or that the eigenvector sensitivity equations were somehow misapplied in this analysis. An eigenvector sensitivity for the multiparameter eigenvalue form of the power flow equations is therefore an open question for future research.

The multiparameter eigenvalue formulation of the power flow equations enables future advances in eigenvalue theory to benefit power systems research. Further developments in direct solution techniques for multiparameter eigenvalue problems may facilitate the solution of the power flow equations for power systems with more than two buses. However, it is important to note that much of the existing multiparameter eigenvalue research has focused on limited problems, such as restrictions to Hermitian matrices or definiteness requirements. The matrices resulting from the power flow equations do not have these properties (the only evident special property is that the eigenvectors and eigenvalues are real). Therefore, future developments in direct solution techniques for multiparameter eigenvalue problems must not be dependent on special matrix properties in order to be applicable to the power flow equations.

The multiparameter eigenvalue formulation of the power flow equations may still prove useful even if relevant direct solution techniques are not developed. For instance, discovery of an upper bound to the number of real solutions of a multiparameter eigenvalue problem would be useful for the continuation power flow method [14],[6]. Given an initial solution, the continuation power flow finds all solutions to the power flow equations by tracing from one solution to the next. The tracing method varies a single parameter, such as the real power injection at a bus, until a new solution is found. Ensuring that all solutions are found may require the continuation power flow to continue tracing after actually finding all solutions. An upper bound on the number of real solutions provided by future advances in multiparameter eigenvalue theory would provide the continuation power flow with a stop condition to prevent unnecessary tracing.

The future discovery of conditions for the existence of solutions to multiparameter eigenvalue problems is an additional advancement in multiparameter eigenvalue theory that has the potential for practical application to the power flow equations. The power flow equations for some operating conditions, particularly when the system is heavily loaded, may not have any real valued solutions (since the power flow equations are derived by separating the real and imaginary parts of the complex power injections, physically meaningful solutions for the voltage components must be real). On the other hand, solutions to solvable systems may not be found due to convergence problems inherent to numeric solution codes. If a solution is not found, engineers may be uncertain of whether there truly are no solutions or if the numeric code is not converging to a solution that does in fact exist. Conditions for the existence of a solution to multiparameter eigenvalue problems would eliminate this uncertainty. Also, for some purposes, such as determining the point of voltage collapse, the details of a solution may not be particularly important; it may be sufficient to simply determine whether any real solutions exist. With future research into conditions for the existence of real solutions, the multiparameter eigenvalue formulation of the power flow equations could play a role for these purposes as well.

The multiparameter eigenvalue formulation of the power flow equations also allows for power flow research to advance multiparameter eigenvalue theory. For instance, it is possible that a method similar to the continuation power flow could be applied to multiparameter eigenvalue problems. Starting from an initial solution to the multiparameter eigenvalue problem, it is possible that another solution could be found by varying a single entry in one of the matrices. An analogous method to the continuation power flow may be able to find all real solutions to a multiparameter eigenvalue problem by varying each matrix entry. This is an open question for future research.

Chapter 4

Conclusion

This thesis has presented eigenvalue formulations for two power systems models: the induction machine initial conditions problem, which is formulated as a standard eigenvalue problem, and the power flow equations, which are formulated as a multiparameter eigenvalue problem. The eigenvalue formulations incorporate the multiplicative nonlinearities that are inherent in both of these models.

Chapter 1 contains an overview of both models and a review of standard, generalized, and multiparameter eigenvalue theory. The eigenvalue review includes a method for solving two parameter eigenvalue problems and the equations for eigenvalue sensitivity analysis of multiparameter eigenvalue problems.

Chapter 2 introduces the induction machine initial conditions problem. This problem involves the determination of the stator and rotor currents, rotor speed, and mechanical torque of an induction machine from known values of real power input and stator voltage magnitude and angle. This problem must be solved when performing dynamic simulations of power systems to initialize induction machine internal variables. The real power, voltage magnitude, and voltage angle values can be obtained from solution of the power flow equations. Finding the induction machine initial conditions for the dynamic simulation requires determining the steady state operating point. Steady state equations derived from the induction machine dynamic equations are converted to standard eigenvalue form using the conditions imposed at the machine terminals. In the standard eigenvalue form, the rotor speed can be obtained from the eigenvalue and the stator and rotor currents can be obtained from the eigenvector. The torque is a function of the currents and can therefore be directly determined after solving the eigenvalue formulation. In contrast to existing iterative methods, the eigenvalue formulation has the advantage of finding all solutions, stable and unstable, to the induction machine initial conditions problem and can easily determine when no solutions exist. The eigenvalue formulation is developed for both single-cage and double-cage induction machines. Numeric examples are provided for both single-cage and double-cage machines.

Chapter 3 explores a multiparameter eigenvalue formulation of the power flow equations. After introducing this formulation, the chapter develops expressions for the left and right eigenvectors. Next, the chapter details a method for solving the two parameter eigenvalue formulation for two bus systems with both PV and PQ buses. This method converts the two parameter eigenvalue formulation into a set of generalized eigenvalue problems that must be simultaneously satisfied. Next, motivated by eigenvalue problem structure, the multiparameter eigenvalue formulation for an arbitrarily sized system was converted into a form that resembles a standard eigenvalue problem with a voltage dependent matrix. Although no practical applications of this reformulation of the power flow equations have yet been discovered, it is an interesting theoretical development. Finally, an eigenvalue sensitivity analysis of the multiparameter eigenvalue formulation of the power flow equations was performed. The eigenvalue sensitivity gives equivalent results to the Jacobian linearization of the power flow equations.

The multiparameter eigenvalue formulation of the power flow equations enables advances in multiparameter eigenvalue theory to contribute to power systems engineering knowledge. As described in Section 3.7, specific advances that would be beneficial include the development of general solution techniques for multiparameter eigenvalue problems with more than two parameters, a method for determining the number of real solutions, and conditions for the existence of any real solutions to multiparameter eigenvalue problems. It is also possible that the multiparameter eigenvalue formulation of the power flow equations will enable application of theory and solution techniques for the power flow equations to multiparameter eigenvalue problems. A specific open question of interest is whether an analogous technique to the continuation power flow method could be used to find solutions to other multiparameter eigenvalue problems.

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Appendix A: Solution to the Induction Machine Initial Conditions Problem

$\mathbf{D}^{-1}\mathbf{B}$ for the single-cage induction machine model has rank two. Therefore, the eigenvalue problem (2.14) has a second order characteristic equation. This equation can be solved directly using the quadratic equation rather than using iterative eigenvalue solution techniques. The characteristic equation for $\mathbf{D}^{-1}\mathbf{B}$ is given in (A.1).

$$\det\left(\mathbf{D}^{-1}\mathbf{B}-\lambda\mathbf{I}\right)=m\lambda^2\left(a\lambda^2+b\lambda+c\right)=0 \tag{A.1}$$

where

$$\begin{aligned}
m =&\frac{1}{\omega_s}\Big(2R_s^2X_rX_m+2R_sX_m^2R_r-R_AR_sR_r^2-R_AR_sX_r^2-R_AR_sX_m^2-R_AX_m^2R_r\\
&+2X_s^2X_rX_m+R_s^2R_r^2+R_s^2X_r^2+R_s^2X_m^2+X_s^2R_r^2+X_s^2X_r^2+X_s^2X_m^2\\
&+X_m^2R_r^2+X_m^2X_r^2+2X_sX_mR_r^2+2X_sX_mX_r^2+2X_sX_rX_m^2-2R_AR_sX_rX_m\Big)^{-1}\\
a =&\omega_s^2\Big(X_m^2X_s^2+X_r^2X_m^2+R_r^2X_m^2+R_s^2X_m^2+R_s^2R_r^2+R_s^2X_r^2\\
&-R_sX_m^2R_A+2R_sX_m^2R_r-2R_sX_mX_rR_A-R_sX_r^2R_A-R_sR_AR_r^2+2X_m^2X_sX_r\\
&-X_m^2R_rR_A+2X_mX_r^2X_s+2X_mX_s^2X_r+2X_mX_sR_r^2+X_s^2R_r^2+X_r^2X_s^2+2R_s^2X_mX_r\Big)\\
b =&\omega\Big(-4R_s^2X_rX_m-2R_sX_m^2R_r+2R_AR_sX_r^2+2R_AR_sX_m^2\\
&+R_AX_m^2R_r-4X_s^2X_rX_m-4X_sX_mX_r^2-4X_sX_rX_m^2+4R_AR_sX_rX_m\\
&-2R_s^2X_r^2-2R_s^2X_m^2-2X_s^2X_r^2-2X_s^2X_m^2-2X_m^2X_r^2\Big)\\
c =&2R_s^2X_rX_m-R_AR_sX_r^2-R_AR_sX_m^2+2X_s^2X_rX_m+R_s^2X_r^2+R_s^2X_m^2\\
&+X_s^2X_r^2+X_s^2X_m^2+X_m^2X_r^2+2X_sX_mX_r^2+2X_sX_rX_m^2-2R_AR_sX_rX_m
\end{aligned}$$

Similarly, $\mathbf{D}^{-1}\mathbf{B}$ for the double-cage induction machine model has rank four. Therefore, the eigenvalue problem (2.14) has a fourth order characteristic equation. This equation can also be solved directly using the equation for quartic equations given in [20].