Identifying Redundant Flow Limits on Parallel Lines

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***Abstract*—Many power system optimization problems constrain line flows with limits specified in terms of the magnitudes of apparent power or current flows. The set of line-flow constraints may have redundancies (i.e., the feasible space may be unchanged upon removal of a subset of the line-flow constraints), which unnecessarily complicate optimization problems. This letter describes an algorithm for identifying redundant line-flow constraints corresponding to certain parallel lines. After formulating the constraints as ellipsoids in the voltage variables, redundancies are detected from the absence of intersections between pairs of ellipsoids corresponding to parallel lines. This algorithm is demonstrated using several large test cases.**

***Index Terms*—Line-flow limits, Power system optimization**

I. Introduction

VARIOUS optimization problems are used to design and operate electric power systems [1]. These problems typically require the satisfaction of line-flow limits specified in terms of the magnitudes of apparent power or current flows. The set of line-flow constraints may contain redundancies such that removal of some constraints does not change the problem's feasible space. Elimination of these redundant constraints simplifies optimization problems. Compared to linear programming, which has a variety of preprocessing methods [2], [3], identifying redundant constraints in nonlinear programs is generally more difficult [4].

This letter presents an algorithm for identifying certain redundant line-flow constraints associated with parallel lines. Section II formulates line-flow constraints on apparent power and current magnitude as ellipsoids and then presents an algorithm that identifies redundancies among line-flow constraints on parallel lines. This algorithm is adopted from the approach in [5] for detecting the intersection of ellipsoids. Section III demonstrates the proposed algorithm using several test cases.

II. Redundancy Identification Algorithm

Consider the transmission line model shown in Fig. 1 from bus l to bus m with an ideal transformer that has a specified turns ratio $\tau_{lm}e^{\mathbf{j}\phi_{lm}} : 1$ in series with a Π circuit with series impedance $R_{lm} + \mathbf{j}X_{lm}$ (equivalent to an admittance of $g_{lm} + \mathbf{j}b_{lm} = \frac{1}{R_{lm}+\mathbf{j}X_{lm}}$) and shunt admittance $\mathbf{j}b_{sh,lm}$, where $\mathbf{j} = \sqrt{-1}$. Define the terminal voltage phasors as $V_l = V_{dl} + \mathbf{j}V_{ql}$ and $V_m = V_{dm} + \mathbf{j}V_{qm}$. This section initially considers flow limits specified in terms of apparent power flows, $S_{lm} := |V_l|\,|I_{lm}|$ and $S_{ml} := |V_m|\,|I_{ml}|$, where $|\cdot|$ indicates the magnitude of a complex scalar. Let $\operatorname{svd}(\,\cdot\,)$ and $\operatorname{eig}(\,\cdot\,)$ denote the singular values and eigenvalues of a matrix, $|\cdot|_0$ the number of non-zero elements, $|\cdot|_2$ the two-norm, $(\,\cdot\,)^\dagger$ the pseudoinverse, and $(\,\cdot\,) \succeq 0$ positive semidefiniteness.

Consider the flow constraints on a pair of parallel lines, which necessarily share the same terminal voltage phasors V_l and V_m. For the first line (resp. second), denote the current flows into the l and m buses as $\hat{I}_{lm}$ and $\hat{I}_{ml}$ (resp. $\tilde{I}_{lm}$ and $\tilde{I}_{ml}$) and the apparent power flow limit as $\hat{S}_{lm}^{\max}$ (resp. $\tilde{S}_{lm}^{\max}$).

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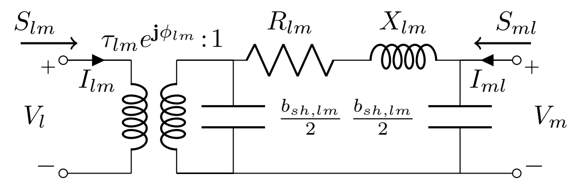

Fig. 1. Line Model.

The constraints $\hat{S}_{lm} \leq \hat{S}_{lm}^{\max}$ and $\hat{S}_{ml} \leq \hat{S}_{lm}^{\max}$ are redundant (implied by $\tilde{S}_{lm} \leq \tilde{S}_{lm}^{\max}$ and $\tilde{S}_{ml} \leq \tilde{S}_{lm}^{\max}$) if

$$\frac{\hat{S}_{lm}}{\hat{S}_{lm}^{\max}} \leq \frac{\tilde{S}_{lm}}{\tilde{S}_{lm}^{\max}} \text{ and } \frac{\hat{S}_{ml}}{\hat{S}_{lm}^{\max}} \leq \frac{\tilde{S}_{ml}}{\tilde{S}_{lm}^{\max}}, \qquad \forall\, V_l,\, V_m \in \mathbb{C}. \tag{1a}$$

Conversely, $\tilde{S}_{lm} \leq \tilde{S}_{lm}^{\max}$ and $\tilde{S}_{ml} \leq \tilde{S}_{lm}^{\max}$ are redundant if

$$\frac{\hat{S}_{lm}}{\hat{S}_{lm}^{\max}} > \frac{\tilde{S}_{lm}}{\tilde{S}_{lm}^{\max}} \text{ and } \frac{\hat{S}_{ml}}{\hat{S}_{lm}^{\max}} > \frac{\tilde{S}_{ml}}{\tilde{S}_{lm}^{\max}}, \qquad \forall\, V_l,\, V_m \in \mathbb{C}. \tag{1b}$$

Otherwise, neither set of constraints is identified as redundant.[1]

Since $|V_l|\,, |V_m| > 0$, the conditions on apparent power flows (1) for parallel lines are equivalent to corresponding inequalities on squared-magnitudes of current flows. That is, the first line-flow constraint is redundant if

$$\frac{\left|\hat{I}_{lm}\right|^2}{\left(\hat{S}_{lm}^{\max}\right)^2} \leq \frac{\left|\tilde{I}_{lm}\right|^2}{\left(\tilde{S}_{lm}^{\max}\right)^2} \text{ and } \frac{\left|\hat{I}_{ml}\right|^2}{\left(\hat{S}_{lm}^{\max}\right)^2} \leq \frac{\left|\tilde{I}_{ml}\right|^2}{\left(\tilde{S}_{lm}^{\max}\right)^2},\ \forall\, V_l,\, V_m \in \mathbb{C}, \tag{2a}$$

and the second line-flow constraint is redundant if

$$\frac{\left|\hat{I}_{lm}\right|^2}{\left(\hat{S}_{lm}^{\max}\right)^2} > \frac{\left|\tilde{I}_{lm}\right|^2}{\left(\tilde{S}_{lm}^{\max}\right)^2} \text{ and } \frac{\left|\hat{I}_{ml}\right|^2}{\left(\hat{S}_{lm}^{\max}\right)^2} > \frac{\left|\tilde{I}_{ml}\right|^2}{\left(\tilde{S}_{lm}^{\max}\right)^2},\ \forall\, V_l,\, V_m \in \mathbb{C}. \tag{2b}$$

Define $x = \begin{bmatrix} V_{dl} & V_{dm} & V_{ql} & V_{qm} \end{bmatrix}^\intercal$, where $(\,\cdot\,)^\intercal$ denotes the transpose. In terms of the voltage components in x,

$$|I_{lm}|^2 / \left(S_{lm}^{\max}\right)^2 = x^\intercal\, \mathbf{M}_{lm}\, x, \tag{3a}$$

$$|I_{ml}|^2 / \left(S_{lm}^{\max}\right)^2 = x^\intercal\, \mathbf{M}_{ml}\, x, \tag{3b}$$

where

$$\mathbf{M}_{lm} := \begin{bmatrix} k_1/\tau_{lm}^4 & -k_3/2 & 0 & k_4/2 \\ -k_3/2 & k_2 & -k_4/2 & 0 \\ 0 & -k_4/2 & k_1/\tau_{lm}^4 & -k_3/2 \\ k_4/2 & 0 & -k_3/2 & k_2 \end{bmatrix} / \left(S_{lm}^{\max}\right)^2, \tag{4a}$$

$$\mathbf{M}_{ml} := \begin{bmatrix} k_2 & -k_5/2 & 0 & k_6/2 \\ -k_5/2 & k_1 & -k_6/2 & 0 \\ 0 & -k_6/2 & k_2 & -k_5/2 \\ k_6/2 & 0 & -k_5/2 & k_1 \end{bmatrix} / \left(S_{lm}^{\max}\right)^2, \tag{4b}$$

$$k_0 := 2b_{lm}^2 + b_{sh,lm}b_{lm} + 2g_{lm}^2, \tag{4c}$$

$$k_1 := b_{lm}^2 + b_{lm}b_{sh,lm} + b_{sh,lm}^2/4 + g_{lm}^2, \tag{4d}$$

$$k_2 := \left(b_{lm}^2 + g_{lm}^2\right)/\tau_{lm}^2, \tag{4e}$$

[1]The conditions in (1) check whether the flows on one line are less than the flows on the other line (relative to their limits) for all terminal voltages.

$$k_3 := \left(k_0 \cos\phi_{lm} + b_{sh,lm} g_{lm} \sin\phi_{lm}\right)/\tau_{lm}^3, \tag{4f}$$
$$k_4 := \left(k_0 \sin\phi_{lm} - b_{sh,lm} g_{lm} \cos\phi_{lm}\right)/\tau_{lm}^3, \tag{4g}$$
$$k_5 := \left(k_0 \cos\phi_{lm} - b_{sh,lm} g_{lm} \sin\phi_{lm}\right)/\tau_{lm}, \tag{4h}$$
$$k_6 := \left(k_0 \sin\phi_{lm} + b_{sh,lm} g_{lm} \cos\phi_{lm}\right)/\tau_{lm}. \tag{4i}$$

To evaluate the conditions in (2), consider the constraints $x^\intercal \mathbf{M}_{lm} x \leq 1$ and $x^\intercal \mathbf{M}_{ml} x \leq 1$. The feasible spaces defined by $x^\intercal \mathbf{M}_{lm} x \leq 1$ and $x^\intercal \mathbf{M}_{ml} x \leq 1$ are the interiors of four-dimensional ellipsoids centered at the origin.[2] Substituting (3) into (2) reveals that identification of redundancies among constraints on parallel lines is equivalent to determining whether the ellipsoids associated with the first line, $x^\intercal \hat{\mathbf{M}}_{lm} x \leq 1$ and $x^\intercal \hat{\mathbf{M}}_{ml} x \leq 1$, are contained within the ellipsoids associated with the second line, $x^\intercal \tilde{\mathbf{M}}_{lm} x \leq 1$ and $x^\intercal \tilde{\mathbf{M}}_{ml} x \leq 1$, where $(\tilde{\cdot})$ and $(\hat{\cdot})$ denote quantities associated with each line. With shared terminal voltages V_l and V_m, note that ellipsoids associated with parallel lines have a common center. Also note that substituting a maximum current limit $I_{lm}^{\max}$ for $S_{lm}^{\max}$ enables identification of redundancies among line-flow limits specified in terms of current magnitudes.

The key step in detecting redundant flow constraints is the ellipsoidal containment identification approach in Alg. 1, which is extended from [5]. Alg. 1 considers a pair of generic ellipsoids, $x^\intercal \mathbf{A} x \leq 1$ and $x^\intercal \mathbf{B} x \leq 1$, first testing the special cases of identical ellipsoids (Step 1) and degenerate ellipsoids, i.e., cylinders (Steps 2–5). Degenerate ellipsoids are identified via zero eigenvalues λ_i^A or λ_i^B. Steps 2–5 evaluate whether the cylindric dimensions of the degenerate ellipsoids are mutually aligned. If so, or if the ellipsoids are non-degenerate, Alg. 1 applies a coordinate transformation to convert the non-degenerate dimensions of the first ellipsoid into a sphere with unity-length radius. The reciprocals of the eigenvalues associated with the matrix for the second ellipsoid in the transformed coordinate system (Steps 6–7) give the squared lengths of the transformed ellipsoid's semi-axes. If all non-degenerate semi-axes of the second ellipsoid have length greater than one (i.e., all associated eigenvalues are less than one), then the second ellipsoid completely contains the first (Steps 8, 9), and vice-versa (Steps 8, 10). Otherwise, neither ellipsoid contains the other (Step 11).

Alg. 2 applies Alg. 1 to the ellipsoids associated with both terminals of each pair of parallel lines. If the ellipsoids associated with both terminals of one line contain those of the other line, Alg. 2 identifies the flow constraint on the former line as redundant. Note that the eigendecompositions of the 4×4 matrices passed to Alg. 1 can be quickly computed analytically via the formula for the roots of a quartic.

Algorithm 1 Check Ellipsoidal Containment

In: $\mathbf{A} \succeq 0$, $\mathbf{B} \succeq 0$ with eigenvalues λ_i^A, λ_i^B, eigenvectors η_i^A, η_i^B.
Out: $\mu_A = 1$ if $x^\intercal \mathbf{A} x \leq 1$ contains $x^\intercal \mathbf{B} x \leq 1$, else 0.
$\mu_B = 1$ if $x^\intercal \mathbf{B} x \leq 1$ contains $x^\intercal \mathbf{A} x \leq 1$, else 0.

1: **if** $\mathbf{A} = \mathbf{B}$, return $\mu_A = 1$ and $\mu_B = 1$ (i.e., identical ellipsoids).
2: $\mathcal{Z}_A = \{i \mid \lambda_i^A = 0\}$, $\mathcal{Z}_B = \{i \mid \lambda_i^B = 0\}$, $\sigma = \text{svd}\left(\left[\eta_{\mathcal{Z}_A}^A,\, \eta_{\mathcal{Z}_B}^B\right]\right)$.
3: **if** $|\mathcal{Z}_B|_0 > |\mathcal{Z}_A|_0$ and $|\sigma|_0 > |\mathcal{Z}_B|_0$, return $\mu_A = 0$, $\mu_B = 0$.
4: **if** $|\mathcal{Z}_A|_0 > |\mathcal{Z}_B|_0$ and $|\sigma|_0 > |\mathcal{Z}_A|_0$, return $\mu_A = 0$, $\mu_B = 0$.
5: **if** $|\mathcal{Z}_A|_0 = |\mathcal{Z}_B|_0 > 0$ and $|\sigma|_0 \neq |\mathcal{Z}_A|_0$, return $\mu_A = 0$, $\mu_B = 0$.
6: $\mathbf{D} = \text{diag}([\sqrt{\lambda_1^A}\, \left|\eta_1^A\right|_2, \ldots, \sqrt{\lambda_4^A}\, \left|\eta_4^A\right|_2])^\dagger$.
7: $\lambda^{B'} = \text{eig}(\mathbf{D}[\eta_1^A, \ldots, \eta_4^A]^\intercal \mathbf{B} [\eta_1^A, \ldots, \eta_4^A] \mathbf{D})$.
8: $\mathcal{Z}_{\neq 0}^{B'} = \{i \mid \lambda^{B'} \neq 0\}$, $\mathcal{Z}_{\geq 1}^{B'} = \{i \mid \lambda^{B'} \geq 1\}$, $\mathcal{Z}_{<1}^{B'} = \{i \mid \lambda^{B'} < 1\} \cap \mathcal{Z}_{\neq 0}^{B'}$.
9: **if** $|\mathcal{Z}_B|_0 \geq |\mathcal{Z}_A|_0$ and $|\mathcal{Z}_{<1}^{B'}|_0 = |\mathcal{Z}_{\neq 0}^{B'}|_0$, return $\mu_A = 0$, $\mu_B = 1$.
10: **if** $|\mathcal{Z}_A|_0 \geq |\mathcal{Z}_B|_0$ and $|\mathcal{Z}_{\geq 1}^{B'}|_0 = |\mathcal{Z}_{\neq 0}^{B'}|_0$, return $\mu_A = 1$, $\mu_B = 0$.
11: **Otherwise** return $\mu_A = 0$, $\mu_B = 0$.

Algorithm 2 Redundant Line Flow Constraint Identification

1: **for each** pair of parallel lines, $\tilde{\mathbf{M}}_{lm}$, $\tilde{\mathbf{M}}_{ml}$ and $\hat{\mathbf{M}}_{lm}$, $\hat{\mathbf{M}}_{ml}$ **do**
2: Alg. 1 with $\mathbf{A} = \tilde{\mathbf{M}}_{lm}, \mathbf{B} = \hat{\mathbf{M}}_{lm}$, yielding $\tilde{\mu}_{lm} = \mu_A, \hat{\mu}_{lm} = \mu_B$.
3: Alg. 1 with $\mathbf{A} = \tilde{\mathbf{M}}_{ml}, \mathbf{B} = \hat{\mathbf{M}}_{ml}$, yielding $\tilde{\mu}_{ml} = \mu_A, \hat{\mu}_{ml} = \mu_B$.
4: **if** $\tilde{\mu}_{lm} = 1$ and $\tilde{\mu}_{ml} = 1$, $x^\intercal \hat{\mathbf{M}} x \leq 1$ is redundant.
5: **else if** $\hat{\mu}_{lm} = 1$ and $\tilde{\mu}_{ml} = 1$, $x^\intercal \tilde{\mathbf{M}} x \leq 1$ is redundant.
6: **else** neither constraint identified as redundant.

TABLE I
REDUNDANT LINE-FLOW LIMITS

Case Name	Num. Lines	Num. Parallel Lines	Num. Redundant Limits
PL-2383wp	2896	20	6
PL-2736sp	3269	12	4
PL-2737sop	3269	12	4
PL-2746wop	3307	16	5
PL-2746wp	3279	12	4
PL-3012wp	3572	12	5
PL-3120sp	3693	18	8
PL-3375wp	4161	178	5
PEGASE-89	210	8	2
PEGASE-1354	1991	519	203
PEGASE-2869	4582	1157	316
PEGASE-9241	16049	3503	650

III. Numeric Results

Table I shows the number of redundant apparent power flow limits identified for several large test cases from [6], [7]. For several of the PEGASE test cases [7], parallel lines accounted for over 25% of the total number of lines, and over 25% of the flow limits on the parallel lines are redundant.

The benefits from Alg. 2 depend on the chosen solver. As one example, the MIPS solver in MATPOWER [6] is 5.7%, 2.0%, and 3.2% faster for the optimal power flow problems corresponding to the PEGASE 1354-, 2869-, and 9241-bus test cases, respectively, after applying Alg. 2.

[2]The function $f(I) := |I|^2$ is convex. Since the currents I_{lm} and I_{ml} are linear transformations of the voltages, which preserve convexity, (3) is convex. A convex inequality of the form $x^\intercal \mathbf{M} x \leq 1$ has an ellipsoidal feasible space.

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