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Sufficient Conditions for Power Flow Insolvability Considering Reactive Power Limited Generators with Applications to Voltage Stability Margins

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Abstract

For the non-linear power flow problem with PQ and reactive power limited slack and PV buses, we present two sufficient conditions under which the specified set of non-linear algebraic equations has no solution. The first condition uses a semidefinite programming relaxation of the power flow equations along with binary variables to model the generators' reactive power capabilities. As a byproduct, this condition yields a voltage stability margin to the power flow solvability boundary. The second condition formulates the power flow equations, including generator reactive power limits, as a system of polynomials and uses real algebraic geometry and sum of squares programming to create infeasibility certificates which prove power flow insolvability.

Introduction

Power flow studies are the cornerstone of power system analysis and design. They are used in planning, operation, economic scheduling, transient stability, and contingency studies. The power flow equations model the relationship between voltages and active and reactive power injections in a power system. The non-linear power flow equations may not have any solutions (the power flow equations are said to be insolvable). That is, it is possible to choose a set of power injections for which no valid corresponding voltage profile exists. It is also possible that no power flow solutions have reactive power injections that can be supported by the generators. That is, enforcing reactive power limits may result in power flow insolvability within the generators' capabilities [1]–[3]. Practical cases that may be insolvable include long-range planning studies for which the studied system may not be able to support projected loads and contingency studies for which the loss of one or more components may yield a network configuration that is similarly inoperable for the specified injections.

This paper presents two sufficient conditions that, when satisfied, rigorously classify a specified case as insolvable. The first condition uses mixed-integer semidefinite programming and yields a voltage stability margin that characterizes a distance to the power flow solvability boundary [4]. The second condition uses real algebraic geometry and sum of squares programming [5] to generate infeasibility certificates which prove power flow insolvability.

In engineering practice, large-scale non-linear power flow equations are typically solved using iterative numerical techniques, most commonly Newton-Raphson or its variants [6]. These rely on an initial guess of the solution voltage magnitudes and angles and are only locally convergent. They generally do not converge to a particular solution from an arbitrary initial guess and may show very high sensitivity and highly complex behavior with respect to initial conditions for certain study cases. It is well recognized that the power flow equations may generally have a very large number of solutions; for example, the work of [7] establishes cases for which the number of solutions grows faster than polynomial with respect to network size. For cases having multiple solutions, each solution has a set of initial conditions that converges to that solution in Newton-Raphson iteration. Characterization of Newton-Raphson regions of attraction was the subject of [8], which demonstrated cases for which the boundaries of these attractive sets were factual in nature. So despite the fact that very large-scale problems (10's or 100's of thousands of unknowns) are solved in power engineering practice, as parameters move outside of routine operating ranges the behavior of these equations can be highly complex. Convergence failure for a Newton-Raphson-based commercial software package is far from a reliable indication that no solution exists.

The properties of the Newton-Raphson iteration guarantee (under suitable differentiability assumptions) that the iteration must converge to a solution for an initial condition selected in a sufficiently small neighborhood about that solution [9]. However, when a selected initial condition (or some set of multiple initial conditions) fails to yield convergence, the user of a Newton-Raphson-based software package is left with an indeterminate outcome: does the specified problem have no solution, or has the initial condition(s) simply failed to fall within the attractive set of a solution that does exist?

Development of conditions guaranteeing power flow solution existence has been an active topic of study. For example, [10] describes sufficient conditions for power flow solution existence. However, as sufficient conditions, these are often conservative: a solution may exist for a much larger range of operating points than satisfy the sufficient conditions. Other work on sufficient conditions for power flow solvability includes [11], which focuses on the decoupled (active power-voltage angle, reactive power-voltage magnitude) power flow model. Reference [12] describes a modified Newton-Raphson

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iteration tailored to the type of ill-conditioning that can appear in power systems problems. In more recent work, [13] provides two necessary conditions for saddle-node bifurcation based on lines reaching their static transfer stability limits; however, this work does not yet provide a test for power flow solvability or define a distance to the power flow solvability boundary. Further, these papers do not consider generators with reactive power limits; power flow equations identified as solvable under the conditions proposed in these works may not have any solutions within the generators' reactive power capabilities.

A measure of the distance to the solvability boundary (the set of operating points where a solution exists, but small perturbations may result in the insolvability of the power flow equations [4]) is desirable to ensure that power systems are operated with security margins. If a solution does not exist for a specified set of power injections, a measure of the distance to the solvability boundary indicates how close the power flow equations are to having a solution. If a power flow solution exists, desired margins indicate distances to solution non-existence at the solvability boundary. Existing techniques for calculating such margins rely almost universally on Newton-based, local solution methods. For instance, [14] and [15] use a Newton-Raphson optimal multiplier approach to find the voltage profile that yields the closest power injections to those specified. For solvable sets of power injections, iterative techniques for finding load margins comprised of the locally optimal minimum distance to the power flow solvability boundary are detailed in [16] and [17]. Other approaches use continuation and/or non-linear optimization to calculate a locally optimal minimum load shedding necessary for power flow solvability [18]–[23].

Ideal voltage sources with no limits on reactive power output often serve as simple generator models. However, reactive power limits are relevant to power flow solvability since non-existence of power flow solutions may result from limit-induced bifurcations [1]–[3].

Recognizing the importance of reactive power limits, common industry practice determines static voltage stability margins using repeated power flow calculations to find the "nose point" of a power versus voltage ("P-V") curve while monitoring "reactive margins" on generators (i.e., the margin between the generator's reactive power output at a given operating point and its maximum reactive output). Descriptions of relevant industry standards can be found in such works as [24]–[26].

Previous work by the authors in this area includes a sufficient condition for power flow insolvability that yields voltage stability margins [27]. A semidefinite program is used to evaluate this sufficient condition. In contrast to existing Newton-based methods whose conditions for convergence are inherently local in nature, the semidefinite program in [27] provides a global solution to the optimization problem that is formulated from the originally specified power flow equations. However, the method proposed in [27] has only a rudimentary incorporation of limits on generator reactive power outputs.

In this paper, we present two sufficient conditions under which the power flow equations are guaranteed to be insolvable within the generators' reactive power limits. The first condition is an extension of the work in [27] that uses mixed-integer semidefinite programming (i.e., optimization problems with both integer and semidefinite matrix constraints) to model reactive power limited generators. The ability to achieve a global optimum enables the guarantee of solution non-existence upon satisfaction of a sufficient condition.

The computation for the first condition provides a power injection margin to the power flow solvability boundary. This margin is a non-conservative bound. Thus, for an insolvable set of specified values, a change in power injections by *at least* the amount indicated by the power injection margin is required for the power flow equations to be *potentially* solvable. More precisely, the margin identifies the shortest distance (as measured by the change in power injections in the direction of a specified injection profile) to a point at which the sufficient condition for power flow insolvability fails to be satisfied.

Current mixed-integer semidefinite programming solvers are relatively immature, and unlike algorithms for semidefinite programs, solvers are not assured to run in polynomial time. However, this is an active area of research, and we anticipate that more capable algorithms will become available. Existing tools [28], [29] can solve the proposed formulation for small power system models, and we discuss potential modifications that improve the computational tractability of the proposed formulation with respect to solution algorithms in the literature [30], [31].

The second sufficient condition for power flow insolvability uses the concept of infeasibility certificates from the field of real algebraic geometry [5]. Infeasibility certificates for polynomial equations are calculated using sum of squares decompositions that are themselves computed with semidefinite optimization programs. Specifically, infeasibility certificates use the Positivstellensatz theorem, which states that there exists an algebraic identity to certify the non-existence of real solutions to every infeasible system of polynomial equalities and inequalities [5]. This theorem does not require any assumptions about the system of polynomials. Since the power flow equations can be expressed as a system of polynomial equalities, infeasibility certificates can be directly applied to power flow problems. Further, this paper formulates limits on generator reactive power outputs as a system of polynomial equalities and inequalities and thus provides a means for extending the theory of infeasibility certificates to power flow problems with these limits.

The organization of this paper is as follows. We first give an overview of the power flow equations. We then describe the first sufficient condition for power flow insolvability and define a power injection margin. Next, we provide an overview of infeasibility certificates and sum of squares programming and describe the second proposed sufficient condition. Numeric examples using standard test systems are then provided. We conclude with a discussion of future work.

Power Flow Equations Overview

The power flow equations describe the sinusoidal steady state equilibrium of a power network, and hence are formulated in terms of complex "phasor" representation of circuit quantities (see, for example, Ch. 9 of [32]). The underlying voltage-to-current relationships of the network are linear, but the nature of equipment in a power system is such that injected/demanded complex power at a bus (node) is typically specified, rather than current. The relation of interest is between the active and reactive power injected at each bus and the complex voltages at each bus, and hence the associated equations are non-linear. Using a rectangular representation for complex voltages ($V_i = V_{di} + jV_{qi}$), and rectangular "active/reactive" representation of complex power ($P_i + jQ_i$), the power balance equations at bus i are given by

$$
\begin{aligned}
P_i = f_{Pi}\left(V_d,\, V_q\right) = &V_{di}\sum_{k=1}^{n}\left(\mathbf{G}_{ik}V_{dk} - \mathbf{B}_{ik}V_{qk}\right) \\
&+ V_{qi}\sum_{k=1}^{n}\left(\mathbf{B}_{ik}V_{dk} + \mathbf{G}_{ik}V_{qk}\right) \qquad \text{(1a)}\\
Q_i = f_{Qi}\left(V_d,\, V_q\right) = &V_{di}\sum_{k=1}^{n}\left(-\mathbf{B}_{ik}V_{dk} - \mathbf{G}_{ik}V_{qk}\right) \\
&+ V_{qi}\sum_{k=1}^{n}\left(\mathbf{G}_{ik}V_{dk} - \mathbf{B}_{ik}V_{qk}\right) \qquad \text{(1b)}
\end{aligned}
$$

where $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ is the network admittance matrix and n is the number of buses in the system.

The rectangular voltage components must additionally satisfy the voltage magnitude equation.

$$
V_i^2 = f_{Vi}\left(V_d,\, V_q\right) = V_{di}^2 + V_{qi}^2 \qquad \text{(1c)}
$$

Using the voltage at the slack bus $V_{slack} = V_{d,slack} + jV_{q,slack}$ as an angle reference, $V_{q,slack} = 0$.

To represent typical behavior of equipment in the power system, each bus is classified as PQ, PV, or slack according to the constraints imposed. PQ buses, which typically correspond to loads and are denoted by the set $\mathcal{PQ}$, treat P_i and Q_i as specified quantities and enforce the active power (1a) and reactive power (1b) equations at that bus. PV buses, which typically correspond to generators and are denoted by the set $\mathcal{PV}$, specify a voltage magnitude V_i and active power injection P_i and enforce the active power and voltage magnitude equations (1a) and (1c). The associated reactive power Q_i may be computed as an "output quantity" via (1b). Finally, a single slack bus is selected with specified V_{di} and V_{qi} (typically chosen such that the reference angle is 0°). The set $\mathcal{S}$ denotes the slack bus. The active power P_i and reactive power Q_i at the slack bus are determined from (1a) and (1b); network-wide conservation of complex power is thereby satisfied.

Additionally, generator reactive power outputs must be within specified limits. If a generator's reactive power output is between the upper and lower limits, the generator maintains a constant voltage magnitude at the bus (i.e., the bus behaves like a PV bus). If a generator's reactive power output reaches its upper limit, the reactive power output is fixed at the upper limit and the bus voltage magnitude is allowed to decrease (i.e., the bus behaves like a PQ bus with reactive power injection determined by the upper limit). If the generator's reactive power output reaches its lower limit, the reactive power output is fixed at the lower limit and the voltage magnitude is allowed to increase (i.e., the bus behaves like a PQ bus with reactive power injection determined by the lower limit). Fig. 1 shows the reactive power versus voltage characteristic for this generator model with a voltage setpoint of V^*, lower reactive power limit of Q^{min}, and upper reactive power limit of Q^{max}.

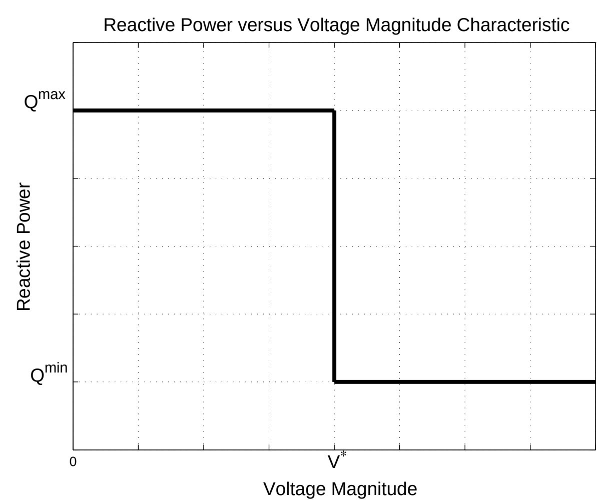

Fig. 1. Reactive Power versus Voltage Magnitude Characteristic

A Sufficient Condition for Power Flow Insolvability Using Mixed-Integer Semidefinite Programming

Mixed-Integer Semidefinite Programming Formulation for a Voltage Stability Margin

This section first formulates a mixed-integer semidefinite program to calculate a voltage stability margin that incorporates generator reactive power limits. Matrices employed in the formulation are defined as

$$\mathbf{Y}_k = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\left(Y_k + Y_k^T\right) & \mathrm{Im}\left(Y_k^T - Y_k\right) \\ \mathrm{Im}\left(Y_k - Y_k^T\right) & \mathrm{Re}\left(Y_k + Y_k^T\right) \end{bmatrix} \tag{2}$$

$$\bar{\mathbf{Y}}_k = -\frac{1}{2}\begin{bmatrix} \mathrm{Im}\left(Y_k + Y_k^T\right) & \mathrm{Re}\left(Y_k - Y_k^T\right) \\ \mathrm{Re}\left(Y_k^T - Y_k\right) & \mathrm{Im}\left(Y_k + Y_k^T\right) \end{bmatrix} \tag{3}$$

$$\mathbf{M}_k = \begin{bmatrix} e_k e_k^T & \mathbf{0} \\ \mathbf{0} & e_k e_k^T \end{bmatrix} \tag{4}$$

where e_k denotes the k^{th} standard basis vector in $\mathbb{R}^n$, the matrix $Y_k = e_k e_k^T \mathbf{Y}$, and superscript T indicates the transpose operator. Notation is adopted from [33]. To write the semidefinite relaxation, first define the vector of voltage coordinates

$$x = \begin{bmatrix} V_{d1} & V_{d2} & \dots & V_{dn} & V_{q1} & V_{q2} & \dots & V_{qn} \end{bmatrix} \tag{5}$$

Then define the rank one matrix

$$\mathbf{W} = x x^T \tag{6}$$

The active and reactive power injections at bus i are then given by $\mathrm{tr}\left(\mathbf{Y}_i \mathbf{W}\right)$ and $\mathrm{tr}\left(\bar{\mathbf{Y}}_i \mathbf{W}\right)$, respectively, where tr indicates the matrix trace operator (i.e., sum of the diagonal elements). The square of the voltage magnitude at bus i is given by $\mathrm{tr}\left(\mathbf{M}_i \mathbf{W}\right)$.

Replacement of the non-convex rank constraint (6) by the less stringent constraint $\mathbf{W} \succeq 0$, where $\succeq 0$ indicates the corresponding matrix is positive semidefinite, yields the convex semidefinite relaxation. This relaxation gives a lower bound for the globally optimal solution of the rank constrained problem. Further, a solution to the semidefinite relaxation has zero duality gap if and only if the rank condition (7) is satisfied (i.e., the relaxation is "tight").

$$\mathrm{rank}\left(\mathbf{W}\right) \leq 2 \tag{7}$$

For a solution with zero duality gap, a unique rank one matrix $\mathbf{W}$ can be recovered by enforcing the known voltage angle at the slack bus [33].

Previous work [27] uses the semidefinite relaxation to define margins to the power flow solvability boundary. The additional flexibility provided by mixed-integer programming is used to extend this work to model reactive power limited generators. The mixed-integer semidefinite programming formulation is

$$\max \ \eta \qquad \text{subject to} \tag{8a}$$

$$\mathrm{tr}\left(\mathbf{Y}_k \mathbf{W}\right) = P_k\, \eta \qquad \forall k \in \{\mathcal{PQ},\, \mathcal{PV}\} \tag{8b}$$

$$\mathrm{tr}\left(\bar{\mathbf{Y}}_k \mathbf{W}\right) = Q_{Dk}\, \eta \qquad \forall k \in \mathcal{PQ} \tag{8c}$$

$$\begin{cases} \mathrm{tr}\left(\bar{\mathbf{Y}}_k \mathbf{W}\right) \geq Q_k^{max} \psi_{Uk} + Q_k^{min}\left(1 - \psi_{Uk}\right) \\ \mathrm{tr}\left(\bar{\mathbf{Y}}_k \mathbf{W}\right) \leq Q_k^{min} \psi_{Lk} + Q_k^{max}\left(1 - \psi_{Lk}\right) \end{cases} \forall k \in \{\mathcal{PV},\, \mathcal{S}\} \tag{8d}$$

$$\begin{cases} \mathrm{tr}\left(\mathbf{M}_k \mathbf{W}\right) \geq \left(V_k^*\right)^2\left(1 - \psi_{Uk}\right) \\ \mathrm{tr}\left(\mathbf{M}_k \mathbf{W}\right) \leq \left(V_k^*\right)^2\left(1 - \psi_{Lk}\right) + d\psi_{Lk} \end{cases} \qquad \forall k \in \{\mathcal{PV},\, \mathcal{S}\} \tag{8e}$$

$$\psi_{Lk} + \psi_{Uk} \leq 1 \qquad \forall k \in \{\mathcal{PV},\, \mathcal{S}\} \tag{8f}$$

$$\sum_{k \in \{\mathcal{PV},\, \mathcal{S}\}} \left(\psi_{Lk} + \psi_{Uk}\right) \leq n_g - 1 \tag{8g}$$

$$\mathbf{W} \succeq 0 \tag{8h}$$

$$\psi_{Uk} \in \{0,\, 1\} \qquad \psi_{Lk} \in \{0,\, 1\} \qquad \forall k \in \{\mathcal{PV},\, \mathcal{S}\} \tag{8i}$$

where d is a large scalar such that the upper limit of (8e) is non-binding when $\psi_{Lk} = 1$ and the scalar n_g is the number of generators (i.e., the number of slack and PV buses). Let η^{max} be a globally optimal solution to (8).

Generator reactive power and voltage magnitude limits are enforced by equations (8d), (8e), (8f), and (8g). When the binary variable ψ_{Uk} is equal to one, the upper reactive power limit of the generator at bus k is binding. Accordingly, (8d) fixes the reactive power output at the upper limit and (8e) sets the lower voltage magnitude limit to zero. When the binary variable ψ_{Lk} is equal to one, the lower reactive power limit of the generator at bus k is binding. Accordingly, (8d) fixes the generator reactive power output at the lower limit and (8e) removes the upper voltage magnitude limit. When both $\psi_{Uk} = 0$ and $\psi_{Lk} = 0$, (8d) constrains the reactive power output within the upper and lower limits and (8e) fixes the voltage magnitude to the specified value V_k^*. Consistency in the reactive power limits is enforced by (8f); a generator's reactive power output cannot simultaneously be at both the upper and lower limits. Finally, reactive power balance is enforced by (8g).

Note that the formulation (8) gives a power injection margin in the specific direction of a uniform, constant-power-factor injection profile; however, the formulation can be extended to consider the impact of non-uniform power injection profiles. Specifically, a semidefinite relaxation can be written for any choice of the right hand side of the power injection constraints (8b) and (8c) that is a linear expression of active and reactive power injections P_k and Q_k, the square of voltage magnitude $(V_k^*)^2$, and the degree-of-freedom η. For instance, with nominal power injections P_{k0} and Q_{k0}, choosing the expressions

$$P_{k0} + \eta \tag{9a}$$

$$Q_{k0} + \tan(\phi_k)\,\eta \tag{9b}$$

for the right hand sides of the active power constraint (8b) and reactive power constraint (8c), respectively, yields an additive power injection margin for the injection profile with specified power factor angles ϕ_k.

Although alternate right-hand-side expressions allow for calculating the power injection margin for non-uniform injection profiles, the insolvability condition that is described next is not applicable for all injection profiles (e.g., a right hand side specifying an injection profile with a non-uniform power factor angle ϕ_k as in (9)).

Optimality Considerations and a Sufficient Condition for Power Flow Insolvability

The solution to (8), η^{max}, is a voltage stability margin to the power flow solvability boundary with consideration of generator reactive power limits. In contrast to traditional iterative methods that may only obtain a locally optimal solution, the formulation (8) yields a globally optimal voltage stability margin.

It is important to note that η^{max} is, in general, a non-conservative bound. Thus, for an insolvable set of specified values, η^{max} indicates the *least* factor by which the power injections must change in the specified profile for the power flow equations to be *potentially* solvable. For a solvable set of specified values, η^{max} indicates the *greatest* factor by which the power injections can change while the power flow equations remain *potentially* solvable.

The non-conservativeness of the bound given by η^{max} is a result of the possibility that a solution to (8) does not satisfy the rank condition of the semidefinite programming relaxation (7) (i.e., the solution to (8) exhibits non-zero duality gap). If a solution to (8) satisfies the rank condition and thus exhibits zero duality gap, a power flow solution can be obtained [33]. This power flow solution is the furthest possible point (i.e., the "nose point") of a P-V curve constructed with consideration of generator reactive power limits. Since (8) can be solved to global optimality, a solution satisfying the rank condition is guaranteed to locate the furthest possible point on the P-V curve. (This is an advantage over traditional iterative approaches which are not guaranteed to locate the furthest possible point.) For solutions satisfying the rank condition (7), the voltage stability margin η^{max} provides the exact distance to the power flow solvability boundary rather than a non-conservative bound.

A globally optimal η^{max} provides a sufficient but not necessary insolvability condition for the power flow equations with generator reactive power limits. Specifically, since η^{max} is a measure of the distance to the power flow solvability boundary,

$$\eta^{max} < 1 \tag{10}$$

is a sufficient but not necessary condition indicating that the specified set of power flow equations has no solution. Conversely,

$$\eta^{max} \geq 1 \tag{11}$$

is a necessary but not sufficient condition for power flow solvability. The conditions (10) and (11) hold regardless of the rank properties of the solution to (8) (i.e., the semidefinite relaxation need not be "tight").

Note that unlike previous work [27] which develops power injection margins using a provably feasible optimization problem, the formulation in (8) does not have a feasibility proof. In other words, it is possible to specify a set of power flow equations for which the optimization problem (8) has an empty feasibility set; the formulation (8) can fail when an injection profile is specified that does not have a value of η such that the power injections have a valid corresponding voltage profile (i.e., the power flow equations are insolvable for any choice of η in (8)).

Computational Considerations

Computational challenges exist in solving mixed-integer semidefinite programs. Without considering the integer constraints, the computational requirements of a semidefinite relaxation of the power flow equations scales with square of the number of buses. Advances in matrix completion decompositions that exploit power system sparsity in semidefinite program relaxations can be applied to ameliorate this challenge [34]–[36]. Thus, each semidefinite program evaluation internal to the mixed-integer semidefinite program solver can be performed significantly more quickly.

The integer constraints introduce added difficulty, and mixed-integer semidefinite programming algorithms are not as mature as, for instance, mixed-integer linear programming algorithms. The existing mixed-integer semidefinite programming solvers BARON [28] and YALMIP [29] are suited for small problems. For instance, YALMIP's branch-and-bound solver is capable of calculating the voltage stability margin using (8) for IEEE test systems [37] with sizes up to 57 buses.

The algorithms described in [30] and [31] claim to be capable of solving large mixed-integer semidefinite programs. The algorithm proposed in [30] is limited by the need to symbolically invert certain submatrices of the positive semidefinite constrained matrix, which is computationally intractable for large matrices. However, this limitation may be overcome for power systems applications by exploiting the sparsity inherent to power system models. Specifically, the matrix completion techniques described in [34]–[36] create a block-diagonal positive-semidefinite-constrained matrix; since each block can be separately inverted, the algorithm described in [30] may be computationally tractable for large power systems.

An additional technique for improving the computational tractability of the proposed method employs a semidefinite relaxation of the integer constraints (8i). This relaxation uses the fact that the binary constraint $\psi \in \{0, 1\}$ is equivalent to the quadratic constraint $\psi^2 - \psi = 0$. Define the constant matrix $\mathbf{N}$ as

$$\mathbf{N} = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \tag{12}$$

If the 2×2 symmetric matrix $\mathbf{R}$ is rank one and $\mathbf{R}^{11} = 1$, then $\mathbf{R}^{22} = \left(\mathbf{R}^{12}\right)^2$, where superscript cd indicates the $(c,\, d)$ entry of the corresponding matrix. Then the equation $\mathrm{tr}\left(\mathbf{NR}\right) = \mathbf{R}^{22} - \mathbf{R}^{12} = 0$ implements the quadratic constraint $\left(\mathbf{R}^{12}\right)^2 - \mathbf{R}^{12} = 0$. For reactive power limited generator bus i, semidefinite relaxations of the quadratic equations (i.e., replacing the requirement $\mathrm{rank}\left(\mathbf{R}\right) = 1$ with the less stringent $\mathbf{R} \succeq 0$) are then implemented with the constraints given in (13), which replace the binary-constraints (8i).

$$\mathrm{tr}\left(\mathbf{NR}_{Ui}\right) = 0 \qquad \mathrm{tr}\left(\mathbf{NR}_{Li}\right) = 0 \tag{13a}$$
$$\mathbf{R}_{Ui}^{11} = 1 \qquad \mathbf{R}_{Li}^{11} = 1 \tag{13b}$$
$$\mathbf{R}_{Ui}^{12} = \psi_{Ui} \qquad \mathbf{R}_{Li}^{12} = \psi_{Li} \tag{13c}$$
$$\mathbf{R}_{Ui} \succeq 0 \qquad \mathbf{R}_{Ui} \succeq 0 \tag{13d}$$

The positive semidefinite constraint (13d) relaxes the rank one requirement on the $\mathbf{R}_{Ui}$ and $\mathbf{R}_{Li}$ matrices. See reference [38] for further discussion on this relaxation technique.

Relaxation of the integer constraints yields an upper bound, denoted as $\bar{\eta}^{max}$, on the distance to the power flow solvability boundary considering reactive power limited generators. Accordingly, the sufficient condition for power flow insolvability (10) holds with this relaxation (i.e., $\bar{\eta}^{max} \leq 1$ is a sufficient condition for power flow insolvability). If the solution to the relaxed problem has rank one $\mathbf{R}_{Li}$ and $\mathbf{R}_{Ui}$ matrices for all reactive power limited generator buses, the semidefinite relaxation of the integer constraints is "tight." With additional satisfaction of the rank condition for $\mathbf{W}$ (7), the proposed formulation gives the exact distance to the power flow solvability boundary.

Unlike the relaxation of the power flow equations, the relaxation of the integer constraints is typically not "tight" and, as will be shown later via numeric examples, may substantially overestimate the distance to the power flow solvability boundary. We therefore propose the following method for obtaining a lower bound on the distance to the power flow solvability boundary. First, calculate $\bar{\eta}^{max}$ with relaxed integer constraints from (13). Then, using the solution to the relaxed problem, set all values of ψ_{Ui} and ψ_{Li} that are over a specified threshold to one with the remainder set to zero. Solve the semidefinite program (8) with the specified values for ψ_{Ui} and ψ_{Li}. If the resulting solution has non-zero duality gap (i.e., the solution satisfies (7)), the solution provides a lower bound, denoted as $\underline{\eta}^{max}$, on the distance to the power flow solvability boundary considering reactive power limited generators. If the rank condition (7) is not satisfied, the solution does not provide a bound on the distance to the power flow solvability boundary.

A Sufficient Condition for Power Flow Insolvability Using Infeasibility Certificates

The second sufficient condition for power flow insolvability uses real algebraic geometry and sum of squares programming to develop infeasibility certificates. After providing an overview of infeasibility certificate theory, we formulate reactive power limits as a system of polynomial inequalities and equalities. This enables application of the Positivstellensatz theorem, which states that there exists an algebraic identity to certify the non-existence of real solutions to every infeasible system of polynomial equalities and inequalities [5].

Overview of Infeasibility Certificate Theory

We first introduce the theory used in constructing infeasibility certificates, specifically the Positivstellensatz theorem and the relationship between sum of squares and semidefinite programming. See [5] for a more detailed overview of this material.

Notation and several definitions are required for understanding the infeasibility certificate theory. This theory applies to a ring of multivariate polynomials with real coefficients, which is denoted as $\mathbb{R}\left[x\right]$ for the variables $\{x_1, \ldots, x_n\}$. Some polynomials have a sum of squares decomposition. These polynomials can be written as

$$p\left(x\right) = \sum_i q_i^2\left(x\right), \qquad q_i \in \mathbb{R}\left[x\right] \tag{14}$$

Note that this decomposition is not necessarily unique. Polynomials with sum of squares decompositions have the important property that they are non-negative for all values of x.

Polynomials with sum of squares decompositions can always be written in the form of a semidefinite program [5]. Define the vector z using monomials of x.

$$z = \begin{bmatrix} 1 & x_1 & x_2 & \dots & x_n & x_1^2 & \dots & x_1 x_n & x_2^2 & \dots \end{bmatrix}^T \tag{15}$$

Then any polynomial with a sum of squares decomposition can be written as

$$p(x) = z^T \mathbf{Q} z \tag{16}$$

where $\mathbf{Q} \succeq 0$. Thus, sum of squares decompositions can be calculated using semidefinite optimization techniques.

Two definitions necessary for creating infeasibility certificates are next introduced. First, the *ideal* of a set of multivariate polynomials $\{f_1, \dots, f_m\}$ is defined as

$$\mathrm{ideal}(f_1, \dots, f_m) = \left\{ f \mid f = \sum_{i=1}^{m} t_i f_i, \qquad t_i \in \mathbb{R}[x] \right\} \tag{17}$$

Note that every polynomial in $\mathrm{ideal}(f_1, \dots, f_m)$ is zero at the zeros of the polynomials $f_1, \dots, f_m$. That is, $f_1(x_0) = 0, \dots, f_m(x_0) = 0$ implies that any polynomial in $\mathrm{ideal}(f_1, \dots, f_m)$ is zero when evaluated at x_0.

Next define the *cone* of the set of multivariate polynomials $\{g_1, \dots, g_r\}$ as

$$\mathrm{cone}(g_1, \dots, g_r) = \left\{ g \mid g = s_0 + \sum_i s_i g_i + \sum_{\{i,j\}} s_{ij} g_i g_j + \sum_{\{i,j,k\}} s_{ijk} g_i g_j g_k + \cdots \right\} \tag{18}$$

where the terms s_0, s_{ij}, s_{ijk}, ... are sum of squares polynomials. Note that every polynomial in $\mathrm{cone}(g_1, \dots, g_r)$ is non-negative if $g_k(x) \geq 0 \; \forall k$.

The Positivstellensatz theorem can then be written as follows. The set of polynomial equations

$$f_i(x) = 0 \qquad i = 1, \dots, m \tag{19a}$$

$$g_k(x) \geq 0 \qquad k = 1, \dots, r \tag{19b}$$

is infeasible in $\mathbb{R}^n$ (i.e., the equations admit no real solution) if and only if there exist polynomials

$$F(x) \in \mathrm{ideal}(f_1, \dots, f_m)$$
$$G(x) \in \mathrm{cone}(g_1, \dots, g_r)$$

such that $F(x) + G(x) = -1$ for all x.

Since F is in $\mathrm{ideal}(f_1, \dots, f_m)$, $F(x_0) = 0$ for any solution x_0 to the equations $f_i(x_0) = 0, \quad i = 1, \dots, m$. Since G is in $\mathrm{cone}(g_1, \dots, g_r)$, $G(y_0) \geq 0$ for any point y_0 in the feasible set of $g_k(y_0) \geq 0, \quad k = 1, \dots, r$. Thus, $F(x_0) + G(x_0)$ must be non-negative for any x_0 that satisfies (19). However, existence of such an x_0 contradicts the fact that $F(x) + G(x) = -1$ for all x. Thus, no valid x_0 exists and the set of equations (19) is infeasible.

Infeasibility Certificates for the Power Flow Equations

Polynomial Formulation of the Power Flow Equations In order to generate infeasibility certificates, we must represent the power flow equations with reactive power limited generators as a system of polynomial inequalities and equalities. The power flow equations without consideration of reactive power limited generators are polynomial equalities in terms of the voltage components V_d and V_q as shown in (1). We next formulate the reactive power limit characteristic shown in Fig. 1 as a set of polynomial equalities and inequalities in the form of (19). Reactive power limits at the generator bus i are formulated as

$$f_{Vi} = (V_i^*)^2 - V_i^- + V_i^+ \tag{20a}$$

$$Q_i^{max} - f_{Qi} = x_i \tag{20b}$$

$$V_i^- x_i = 0 \tag{20c}$$

$$V_i^+ \left(Q_i^{max} - Q_i^{min} - x_i \right) = 0 \tag{20d}$$

$$Q_i^{max} - Q_i^{min} - x_i \geq 0 \tag{20e}$$

$$V_i^+ \geq 0, \quad V_i^- \geq 0, \quad x_i \geq 0 \tag{20f}$$

where the polynomial functions $f_{Qi}(V_d, V_q)$ and $f_{Vi}(V_d, V_q)$ are defined in (1b) and (1c), respectively.

The variable x_i represents the distance to the upper reactive power limit for the generator bus i (i.e., x_i is a "slack variable" for this limit). With x_i constrained to be non-negative in (20f), the reactive power generation at bus i is maintained within its upper limit. Similarly, the distance to the lower reactive power limit is $Q_i^{max} - Q_i^{min} - x_i$, which is constrained to be non-negative in (20e). Reactive power generation is thus greater than or equal to the lower limit. With equality constraints (20a) and (20c), the non-negative variable V_i^- allows the voltage magnitude at bus i to decrease when the reactive power generation is at its upper limit. Similarly, with equality constraints (20a) and (20d), the non-negative variable V_i^+ allows the voltage magnitude at bus i to increase when the reactive power generation is at its lower limit. Thus, the set of equations (20) models the reactive power versus voltage characteristic shown in Fig. 1.

Infeasibility Certificates for the Power Flow Equations Without Considering Reactive Power Limits With a polynomial formulation, infeasibility can be verified using the Positivstellensatz theorem. We first consider the case without reactive power limits on generators (i.e., generators are modeled as ideal voltage sources with fixed voltage V_i^* for any reactive power output). For this case, the power flow equations are entirely in the form of equalities. An infeasibility certificate is found if a polynomial $F\left(V_d, V_q\right)$ in the ideal formed by the power flow equations (1) satisfies

$$F\left(V_d, V_q\right) = -1 \tag{21}$$

A polynomial in the ideal of the power flow equations has the form

$$\begin{aligned} F\left(V_d, V_q\right) = \tau V_{q,slack} + \sum_{i \in \{\mathcal{PV},\mathcal{PQ}\}} \lambda_i \left(f_{Pi} - P_i\right) \\ + \sum_{i \in \mathcal{PQ}} \gamma_i \left(f_{Qi} - Q_i\right) + \sum_{i \in \{\mathcal{S},\mathcal{PV}\}} \mu_i \left(f_{Vi} - V_i^2\right) \end{aligned} \tag{22}$$

where $V_{q,slack}$ is the q-component of the slack bus voltage and τ, λ, γ, and μ are polynomials (which are not necessarily sum of squares) associated with the slack bus angle, active power injection, reactive power injection, and squared voltage magnitude equations, respectively.

Using the Positivstellensatz theorem, the power flow equations are insolvable if there exist polynomials τ, λ, γ, and μ such that $F\left(V_d, V_q\right) = -1$. This condition is evaluated by attempting to find a sum of squares decomposition for the polynomial $-F\left(V_d, V_q\right) - 1$ using semidefinite programming. If such a decomposition exists, the power flow equations are proven insolvable.

This can be understood using the fact that the polynomial $-F\left(V_d, V_q\right) - 1$ is negative for any values of V_d and V_q that are solutions to the power flow equations (1); conversely, a sum of squares decomposition is non-negative for all values of V_d and V_q. Thus, the power flow equations are insolvable if $-F\left(V_d, V_q\right) - 1$ is a sum of squares.

Note that the theory used to develop this result does not provide any information on the necessary degree of the unknown polynomials τ, λ, γ, and μ. A need for high-degree polynomials may make this method computationally intractable, and there are examples of polynomial equations for which high degrees are necessary to prove infeasibility [39]. Fortunately, numerical experience suggests that low-degree choices for τ, λ, γ, and μ often suffice for proving insolvability of the power flow equations. For instance, infeasibility certificates were generated using constant (degree zero) polynomials for the numeric examples provided in this paper.

Infeasibility Certificates for the Power Flow Equations Considering Reactive Power Limits To find infeasibility certificates for the power flow equations with reactive power limited generators (1a), (1b), and (20), form the polynomial

$$\begin{aligned} H\left(V_d, V_q, x, V^+, V^-\right) = \; & \tau V_{q,slack} + \sum_{i \in \{\mathcal{PV},\mathcal{PQ}\}} \lambda_i \left(f_{Pi} - P_i\right) \\ & + \sum_{i \in \mathcal{PQ}} \gamma_i \left(f_{Qi} - Q_i\right) + \sum_{i \in \{\mathcal{S},\mathcal{PV}\}} \Big\{ \psi_{1i} \left(\left(V_i^*\right)^2 - V_i^- + V_i^+ - f_{Vi}\right) \\ & + \psi_{2i}\left(Q_i^{max} - f_{Qi} - x_i\right) + \psi_{3i} V_i^- x_i \\ & + \psi_{4i}\left(Q_i^{max} - Q_i^{min} - x_i\right) V_i^+ + s_{1i}\left(Q_i^{max} - Q_i^{min} - x_i\right) \\ & + s_{2i} V_i^+ + s_{3i} V_i^- + s_{4i} x_i \Big\} \end{aligned} \tag{23}$$

where ψ_{1i}, ψ_{2i}, ψ_{3i}, and ψ_{4i} are polynomials and s_{1i}, s_{2i}, s_{3i}, and s_{4i} are sum of squares polynomials. If the polynomials τ, λ, γ, and ψ and sum of squares polynomials s can be chosen such that $-H\left(V_d, V_q, x, V^+, V^-\right) - 1$ is a sum of squares, the power flow equations with consideration of reactive power limits on generators are insolvable.

As shown in (23), H is a quadratic function of the variables x, V^+, and V^- used to model the reactive power limits as well as the voltage components V_d and V_q. For an n-bus system with n_g reactive power limited generators and constant (degree zero) polynomials chosen for τ, λ, γ, ψ, and s, the number of monomials used in a sum of squares decomposition of H (i.e., the number of entries in z for the form (16)) is equal to $2n + 3n_g + 1$. Since the number of entries in the positive semidefinite matrix $\mathbf{Q}$ in (16) scales as the square of the number of monomials in z, a naïve implementation for creating infeasibility certificates becomes computationally intractable for moderate size systems. However, pre-processing the sum of squares program with the Newton Polytope method [40] decreases the number of monomials required in the decomposition, thus reducing the computational burden of the sum of squares program. Future work includes improving computational tractability; one promising direction is adoption of pre-processing techniques which exploit power system sparsity from applications of semidefinite programming to the optimal power flow problem [34]–[36].

Experience with the IEEE test systems demonstrates that infeasibility certificates are not found with either degree zero or degree one polynomials when both upper and lower limits on generator reactive power outputs limits are modeled. Since the number of monomials required increases combinatorially with the degree chosen for the polynomials, choices of higher degree polynomials are not computationally tractable. However, infeasibility certificates are found by neglecting lower reactive power limits on generator outputs. If lower limits on reactive power outputs are not considered, (20) is simplified by eliminating equations (20d) and (20e) as well as V_i^+ in (20a) and (20f), with corresponding changes to (23). Since lower limits on reactive power outputs are rarely responsible for power flow insolvability through limit-induced bifurcations, neglecting the lower limits is an acceptable approximation for the large majority of cases.

Examples

We next apply the mixed-integer semidefinite programming and the infeasibility certificate formulations to test systems using optimization codes YALMIP [29] and SeDuMi [41]. Consider a power injection profile where the active and reactive injections at both PQ and PV buses are increased at constant power factor as in (8).

We first consider application to the IEEE 14-bus system [37]. The power injection margin calculated from (8) is $\eta^{max} = 1.3522$. Since the solution obtained from (8) satisfies the condition $\mathrm{rank}\left(\mathbf{W}\right) \leq 2$, the condition (11) indicates that a power flow solution exists for power injection changes in the direction of the specified profile up to an injection multiplier of 1.3522. The insolvability condition (10) indicates that no solutions exist for power injection multipliers greater than 1.3522.

Although the IEEE 14-bus system is small enough to find a global optimum to (8) with branch-and-bound techniques, this test case can also illustrate the use of the integer constraint relaxations discussed in this paper. With all $\mathbf{R}_{Ui}$ matrices being rank two, the relaxation of the integer constraints is not "tight." The resulting upper bound $\bar{\eta}^{max}$ of 5.3589 is well above the actual maximum value of 1.3522. In an attempt to obtain a lower bound $\underline{\eta}^{max}$, we set to one all integer variables ψ_{Ui} and ψ_{Li} that are above a threshold of 0.5, with the remainder set to zero. (For this case, all $\psi_{Ui} = 1$ and $\psi_{Li} = 0$ except for the variables corresponding to the slack bus.) The solution to the resulting semidefinite optimization satisfies the rank condition (7) and therefore provides a lower bound $\underline{\eta}^{max}$ of 1.3522. Thus, the lower bound $\underline{\eta}^{max}$ for this case is equal to the actual value of η^{max}.

Considering only upper reactive power limits for computational tractability, an infeasibility certificate is found using (23) with constant (degree zero) polynomials for an injection multiplier of 1.36. This infeasibility certificate proves power flow insolvability for this power injection profile. Note that the infeasibility certificates do not directly provide a measure of the distance to the power flow solvability boundary. However, a measure can be calculated using binary search over loading cases in the direction of the specified power injection profile (uniform power injection changes for these examples).

In Fig. 2, these results are verified by tracing the P-V curve while enforcing generator reactive power limits for the IEEE 14-bus system. When a generator reaches a reactive power limit, the bus is converted to a PQ bus with reactive power injection determined by the binding reactive power limit. The "nose point" of the P-V curve for this system occurs when all generators, including the generator at the slack bus, reach upper reactive power limits. Without the ability to enforce reactive power balance, the power flow solution disappears in a limit-induced bifurcation at a power injection multiplier of 1.3522, thus verifying both of the proposed sufficient conditions for power flow insolvability.

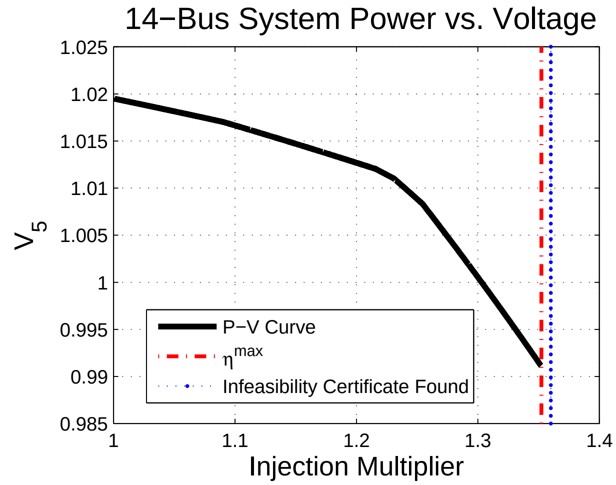

Fig. 2. IEEE 14-bus Power Injection Margin with Generator Reactive Power Limits

System	Trace Nose Point	η^{max}	$\bar{\eta}^{max}$	$\underline{\eta}^{max}$	Infeasibility Certificate
14-bus	1.3522	1.3522	5.3589	1.3522	1.36
30-bus	2.8609	2.8609	3.3218	N/A	2.86
57-bus	1.6486	1.6486	4.4261	1.6486	1.65

TABLE I

STABILITY MARGINS FOR IEEE TEST SYSTEMS CONSIDERING REACTIVE POWER LIMITED GENERATORS

Table I shows the results of the proposed sufficient conditions for several of the IEEE test systems considering reactive power limited generators. The columns of Table I show 1) the system name, 2) the nose point identified by tracing the P-V curve of the high-voltage, stable power flow solution, 3) the value of η^{max} for a global solution to (8) calculated using branch-and-bound techniques, 4) an upper bound $\bar{\eta}^{max}$ resulting from relaxing the integer constraints with (13), 5) a lower bound $\underline{\eta}^{max}$, and 6) the smallest power injection multiplier for which an infeasibility certificate is found using constant (degree zero) polynomials and only upper limits on reactive power generation. A case for which no lower bound $\underline{\eta}^{max}$ could be estimated (i.e., the solution did not satisfy the rank condition (7)) is denoted with "N/A" in the fifth column of Table I.

Note that the only method with a guarantee of the actual distance to the power flow solvability boundary is a global solution to the mixed-integer semidefinite programming formulation (8) that satisfies the rank condition (7). (The rank condition is satisfied by solutions to the IEEE 14, 30, and 57-

bus systems.) The remaining methods provide upper bounds ($\bar{\eta}^{max}$, η^{max} with a solution that does not satisfy the rank condition (7), and infeasibility certificates) and lower bounds (tracing the P-V curve and $\underline{\eta}^{max}$ with a solution that satisfies the rank condition (7)) on the actual distance to the power flow solvability boundary.

The results in Table I verify the proposed sufficient conditions for power flow insolvability. The voltage margin η^{max} from (8) matches the nose points of the P-V curves. Although the upper bound $\bar{\eta}^{max}$ does not give a result close to the nose point, the lower bound $\underline{\eta}^{max}$, when calculable, matches the actual value η^{max}. Finally, infeasibility certificates identify the nose point for each test case.

Infeasibility certificates can also be found without considering reactive power limits on generators. As shown in [42] for the IEEE 118-bus system, there may be loadings for which no power flow solution is found but the sufficient conditions for power flow insolvability are not satisfied. Using (22), the smallest injection multiplier certified infeasible with constant (degree zero) polynomials is equal to the power injection margin calculated using the semidefinite-programming-based sufficient condition for power flow insolvability, which is equivalent to (8) without limits on reactive power generation. (Specifically, while the nose point resulting from a continuation trace of the high-voltage, stable solution is at an injection multiplier of 3.18, neither sufficient condition for insolvability is satisfied until an injection multiplier of 3.27.) This suggests the possibility of a deeper connection between the infeasibility certificates with degree-zero polynomials and the semidefinite-programming-based sufficient condition for power flow insolvability, at least for cases without reactive power limited generators. (Note that computational limitations preclude use of higher-order polynomials, which may more closely identify the nose point.)

Conclusion and Future Work

This paper has presented two sufficient conditions for power flow insolvability considering reactive power limited generators. The first condition formulates a mixed-integer semidefinite program to determine a global voltage stability margin. This margin gives a bound on the distance to the power flow solvability boundary and can be applied to both solvable and insolvable sets of power injections. For solutions that satisfy a rank condition, the proposed formulation gives the exact distance to the solvability boundary (i.e., a guarantee of the "nose point" of the P-V curve). The margin gives a sufficient condition for power flow insolvability with consideration of reactive power limited generators.

The second sufficient condition creates infeasibility certificates to prove power flow insolvability. Writing the power flow equations, including reactive power limits on generators, as a system of polynomial equalities and inequalities allows for application of the Positivstellensatz theorem from the field of real algebraic geometry. If a specified polynomial can be written in sum of squares form, which is determined using semidefinite programming, the power flow equations are proven insolvable.

Both sufficient conditions, along with several approximations to improve computational tractability, are applied to IEEE test systems. The results show that the sufficient conditions are capable of identifying the distance to the power flow solvability boundary.

Future work includes improving the computational tractability of both sufficient conditions in order to apply the proposed methods to large-scale system models. The first sufficient condition may benefit from the application of large-scale mixed-integer semidefinite program algorithms, such as [30] and [31]. For the second sufficient condition, exploiting power system sparsity may allow for use of higher-order polynomials, which may be necessary to prove insolvability for some power flow equations.

Future work also includes the numerous other potential applications for mixed-integer semidefinite programming and real algebraic geometry in the field of electric power systems. Mixed-integer semidefinite programming can be directly applied to problems that have integer constraints (e.g., the unit commitment problem where a power system dispatch is optimized over time with the ability to commit and decommit generators [43] and the optimal transmission switching problem where a generation dispatch and transmission topology is determined to meet a given load [44]). Existing work in this area includes the application of mixed-integer semidefinite programming to the transmission expansion problem [45]. Infeasibility certificates may be applicable to proving insolvability for other power systems problems, such as the optimal power flow and the unit commitment problems.

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