Initializing Dynamic Power System Simulations Using Eigenvalue Formulations of the Induction Machine and Power Flow Models

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***Abstract*—Initializing internal variables in dynamic power system simulations is a two-stage process. First, a power flow model is used to find the steady state bus variables. Second, values for the internal variables associated with bus connected components are determined such that the components' terminal values match the bus variables calculated from the power flow model. Initializing most components' variables is a straightforward, direct process. However, initializing induction machine variables traditionally uses an indirect, iterative process. In this paper, eigenvalue formulations are detailed for both the induction machine initialization and power flow models, which provide a direct method for determining all possible sets of induction machine initializations and offer a novel model for the power flow equations.**

***Index Terms*—Power system simulation, Eigenvalues, Power flow, Induction machines**

I. Introduction

DYNAMIC simulations are essential to engineering analysis of electrical power systems. These simulations are used in long-term planning and design environments and increasingly in operational environments to facilitate flexible power interchanges [1]. Dynamic simulations are required for anticipatory analysis, such as determining voltage and power flow operational limits in order to ensure transient stability, and in post-mortem analysis of blackouts. Power system stability is a major topic in power engineering research [2]–[5].

To initialize a dynamic power system simulation, it is necessary to determine the initial conditions of all devices connected to the system, including synchronous generators, loads, and induction machines. Determining the initial conditions for synchronous generators is straightforward [1]. Load models for dynamic simulations are continually under development. See, for instance, the most recent WECC composite load model, which contains several induction motor models and static voltage dependent components [6]. Initialization of most components in the composite load model is straightforward. Initializing induction machines, however, is a more complex iterative process. Induction machines are particularly important devices since they comprise a significant portion of the load at many buses [7]. For each induction machine connected to a bus, the initial conditions of the machine model's internal state variables must be obtained.

Determining the initial conditions for dynamic power system simulations is a two-stage process: first, a power flow model is used to determine steady state values of the active power (P), the reactive power (Q), the voltage magnitude (V) and the voltage angle (δ) at each bus. Second, the initial conditions for the internal variables of all connected components are set to their steady state values such that the terminal values of the models match the values of P, Q, V, and δ obtained from the power flow model. Since initialization of other components is generally direct and straightforward, we focus on induction machines. Appropriate choice of the internal states of the induction machine model match both the machine's active power and voltage to the bus values that are obtained from the power flow model (i.e. in this initialization approach, machine slip is calculated rather than specified). An additional capacitor is added at the bus to match the reactive power input to the machine with the reactive power obtained from the power flow analysis [7], [8, p. 627].

There are existing solution techniques for both stages of the initial conditions problem. For a textbook treatment on power flow models, including several standard solution techniques (Gauss-Seidel, Newton-Raphson, etc.), see [8]. There has also been significant research attention to the problem of finding all solutions to the power flow model. [9] uses analytic tools from topology and geometry to determine bounds on the number of solutions and the stability of solutions in a lossless network of PV buses. [10] generalizes this analysis to lossy systems of PV buses. Continuation power flow algorithms that can find multiple solutions to the power flow equations were published in [11] and [12]. Other solution techniques for the power flow equations have also been attempted, such as a biologically inspired ant colony algorithm in [13] and a genetic algorithm [14]. To our knowledge, none of the existing literature formulates the power flow model in terms of eigenvalue problems.

A common approach for matching the active power and voltage at the induction machine terminals to the values obtained from the power flow model uses iteration on the machine's slip. This approach is used in such software as Positive Sequence Load Flow Software (PSLF) [15, p. 963], Power System Simulation for Engineering (PSS/E) [16, p. 20-16], and PowerWorld [17]. (Note that other software, such as Electro-Magnetic Transients Program (EMTP) [18, p. 9-19], initializes induction machines by specifying machine slip and calculating active and reactive power demands. This does not require iteration and is straightforward; it is not, therefore, discussed herein.) A potential disadvantage of the iterative approach is that at most one solution is obtained when

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multiple solutions are often possible. For general purposes, the highest speed stable solution is sought; however, for many research purposes, studies can focus on other solutions. For instance, extending work in bifurcation theory [3], [5] to incorporate load models requires multiple solutions to the induction machine initial conditions problem [19].

Our first contribution is a generalized eigenvalue formulation for the induction machine initial conditions model. Assuming linear magnetic relationships between machine fluxes and currents, the steady state electrical equations for the induction machine model are linear in the electric variables and contain a machine speed multiplicative nonlinearity (see the single multiplicative nonlinearity in ω_r in equation (1)). The rotor speed ω_r serves as the eigenvalue in our formulation. The eigenvector is composed of the machine electrical variables (currents in this formulation). Once these are determined, the torque equation is used to initialize the model's mechanical torque model. This paper uses a double-cage induction machine model; a single-cage model is reported in our previous work [20] (double-cage models have been found to be well-suited for aggregate motor load models [21]). This eigenvalue approach has the advantage of providing all solutions, stable and unstable, and can reliably determine when no solutions exist that satisfy the terminal constraints, identified by the absence of finite real eigenvalues.

The eigenvalue formulation has immediate practical value to researchers who study power systems under a wide range of conditions. The advantages of the eigenvalue formulation over traditional iterative methods include 1. Certain research applications, such as determining stability margins and the distance to bifurcation points [3], [5], require the determination of multiple solutions. 2. Finding all solutions ensures that the desired solution can be selected. The single solution obtained from traditional iterative methods does not guarantee that the obtained solution is in fact the desired solution. 3. In contrast to traditional iterative methods, the eigenvalue formulation explicitly indicates when no solutions exist.

Our second contribution is a multiparameter eigenvalue formulation of the power flow model. The power flow model, including active power, reactive power, and voltage magnitude equations, is reformulated as a multiparameter eigenvalue problem. The orthogonal d and q components of the voltages at each bus serve as the eigenvalues in this formulation. The eigenvectors are also composed of these voltage components. The two-parameter formulation of the power flow equations for two-bus systems can be solved directly by decomposing the problem into two generalized eigenvalue problems that must be simultaneously satisfied. Since n-bus systems require $2(n-1)$ parameter eigenvalue problems, which do not yet have a general solution method for $n > 2$, the power flow model for systems with more than two buses is not yet directly solvable from the multiparameter eigenvalue formulation.

Given the limited ability to solve multiparameter eigenvalue problems, this second contribution is mainly of theoretical interest at this point. It does not offer an advance in the ability to solve the power flow equations. Future developments in multiparameter eigenvalue theory may provide additional insights into solutions of the power flow equations. For instance, general solution techniques for multiparameter eigenvalue problems with more than two parameters may enable direct solution of the power flow equations. A method for determining the number of real-valued solutions to multipa-

$$
\begin{bmatrix} V_{ds} \\ V_{qs} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left(\begin{bmatrix} R_s & -(X_{ls}+X_m) & 0 & -X_m & 0 & -X_m \\ (X_{ls}+X_m) & R_s & X_m & 0 & X_m & 0 \\ 0 & -X_m & R_{r1} & -(X_{lr1}+X_m) & 0 & -X_m \\ X_m & 0 & (X_{lr1}+X_m) & R_{r1} & X_m & 0 \\ 0 & -X_m & 0 & -X_m & R_{r2} & -(X_{lr2}+X_m) \\ X_m & 0 & X_m & 0 & (X_{lr2}+X_m) & R_{r2} \end{bmatrix} \right.
$$
$$
\left. + \frac{\omega_r}{\omega_s} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_m & 0 & (X_{lr1}+X_m) & 0 & X_m \\ -X_m & 0 & -(X_{lr1}+X_m) & 0 & -X_m & 0 \\ 0 & X_m & 0 & X_m & 0 & (X_{lr2}+X_m) \\ -X_m & 0 & -X_m & 0 & -(X_{lr2}+X_m) & 0 \end{bmatrix} \right) \begin{bmatrix} I_{ds} \\ I_{qs} \\ I_{dr1} \\ I_{qr1} \\ I_{dr2} \\ I_{qr2} \end{bmatrix}
$$
$$
+ \frac{1}{\omega_s} \begin{bmatrix} (X_{ls}+X_m) & 0 & X_m & 0 & X_m & 0 \\ 0 & (X_{ls}+X_m) & 0 & X_m & 0 & X_m \\ X_m & 0 & (X_{lr1}+X_m) & 0 & X_m & 0 \\ 0 & X_m & 0 & (X_{lr1}+X_m) & 0 & X_m \\ X_m & 0 & X_m & 0 & (X_{lr2}+X_m) & 0 \\ 0 & X_m & 0 & X_m & 0 & (X_{lr2}+X_m) \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I_{ds} \\ I_{qs} \\ I_{dr1} \\ I_{qr1} \\ I_{dr2} \\ I_{qr2} \end{bmatrix} \tag{1}
$$

$$
\frac{2H}{\omega_s} \frac{\mathrm{d}\omega_\mathrm{r}}{\mathrm{d}t} = T_e(I_{ds}, I_{qs}, I_{dr1}, I_{qr1}, I_{dr2}, I_{qr2}) - T_m(\omega_r) \tag{2}
$$

$$
T_e(I_{ds}, I_{qs}, I_{dr1}, I_{qr1}, I_{dr2}, I_{qr2}) = X_m \left(I_{qs}(I_{dr1}+I_{dr2}) - I_{ds}(I_{qr1}+I_{qr2}) \right) \tag{3}
$$

rameter eigenvalue problems would be useful as a stopping condition for continuation power flows [11]. Conditions for the existence of any real solutions to multiparameter eigenvalue problems would be useful for finding the point of voltage collapse and for analyzing power systems in heavily loaded conditions. Such advances await future developments in multiparameter eigenvalue theory. Here we present the formulation and analyze a small system for which existing multiparameter techniques suffice.

We first focus on the induction machine initial conditions model. A dynamic model for a double-cage induction machine adapted from [22] is presented. Next, our eigenvalue formulation for the induction machine initial conditions model is derived and applied to the double-cage machine model. A numeric example and discussion is presented. We then focus on a multiparameter eigenvalue formulation of the power flow model. After presenting this formulation, we derive expressions for the eigenvectors, provide a direct solution of the formulation for two-bus systems, and discuss the potential benefits of this formulation.

II. An Eigenvalue Formulation of the Induction Machines Initial Conditions Model

A. Dynamic Double-Cage Induction Machine Model

A dynamic double-cage induction machine model in the dq frame with linear magnetic relationships and short-circuited rotor windings is adapted from [22]. The double-cage machine model is commonly used to represent the deep-bar effect and is similar to the single-cage induction machine model with an additional rotor circuit branch. The model is given in (1), (2), and (3), where ω_s refers to the electrical excitation frequency and ω_r refers to the rotor speed. X_{ls} is the stator leakage reactance, X_{lr1} and X_{lr2} are the rotor leakage reactances, X_m is the mutual reactance, R_s is the stator resistance, R_{r1} and R_{r2} are the rotor resistances, and H is the inertia constant of the machine and mechanical load. $T_m(\omega_r)$ represents the load torque model, which has parameters whose values must be determined. All quantities are given in per unit representation.

Consistent with common power system simulation packages [15]–[17], machine core losses are neglected. A common representation of core losses involving a linear resistor [23] can easily be incorporated into the induction machine model with no added complexity. However, over a wide range of possible solutions, a linear resistance model of core losses may not be accurate. Neglecting magnetic saturation and core losses is common practice in system-wide studies with induction machine load models [15], [16], [24]. For such large-scale system simulations, the induction motors serve as a coarse representation of aggregate motor load. As this is intended to capture qualitative features of the system response, the details of magnetic saturation and core losses are neglected. More important, from a load modeling perspective, are assumptions of motor mechanical load characteristics. Consequently, saturation and core losses are not typically included in system studies as they would be for detailed simulations of individual machinery. We discuss this further at the end of this section.

The steady state electrical equations for the double-cage induction machine model can be represented in phasor form as shown in (4). The stator current is $I_s = I_{ds} + jI_{qs}$, the rotor currents are $I_{r1} = I_{dr1} + jI_{qr1}$ and $I_{r2} = I_{dr2} + jI_{qr2}$, and the stator voltage is $V_s = V_{ds} + jV_{qs}$.

$$\begin{bmatrix} V_s \\ 0 \\ 0 \end{bmatrix} = \left(\begin{bmatrix} R_s + j(X_{ls} + X_m) & jX_m & jX_m \\ jX_m & R_{r1} + j(X_{lr1} + X_m) & jX_m \\ jX_m & jX_m & R_{r2} + j(X_{lr2} + X_m) \end{bmatrix} + \frac{\omega_r}{\omega_s} \begin{bmatrix} 0 & 0 & 0 \\ -jX_m & -j(X_{lr1} + X_m) & -jX_m \\ -jX_m & -jX_m & -j(X_{lr2} + X_m) \end{bmatrix} \right) \begin{bmatrix} I_s \\ I_{r1} \\ I_{r2} \end{bmatrix} \tag{4}$$

The equivalent circuit representation for (4) is presented in Fig. 1, where $s = 1 - \frac{\omega_r}{\omega_s}$.

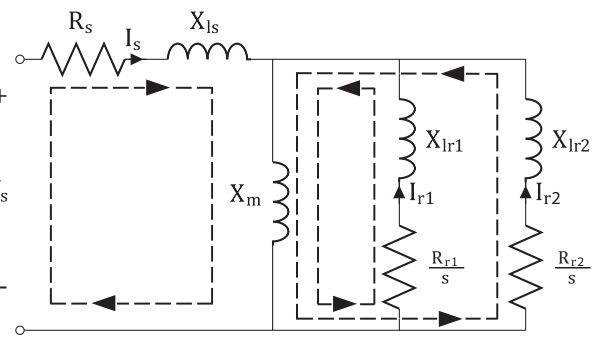

Fig. 1. Double-Cage Induction Machine Steady State Equivalent Circuit

In steady state, shifting the angle of the stator voltage V_s by δ shifts all the currents by the same angle δ. This can be seen by multiplying both sides of (4) by $e^{j\delta}$. Therefore, the input complex power $P + jQ = V_s I_s^*$ is invariant to shifts in the stator voltage angle. We exploit this property in the method to follow.

B. Eigenvalue Formulation of the Initial Conditions Model

The dynamic double-cage induction machine model given by (1), (2), and (3) fits into a more general induction machine dynamic model framework.

$$y = [\mathbf{A} + \omega_r \mathbf{B}]\, x + \mathbf{C}\frac{\mathrm{d}x}{\mathrm{dt}} \tag{5}$$

$$\frac{2H}{\omega_s}\frac{\mathrm{d}\omega_r}{\mathrm{d}t} = T_e(x) - T_m(\omega_r) \tag{6}$$

This model has the applied voltage contained in the vector y and stator and rotor currents contained in the vector x. The rotor windings are short circuited. $\mathbf{A}$ contains all terms that do not depend on the rotational speed ω_r, $\mathbf{B}$ contains all terms that do depend on ω_r, and $\mathbf{C}$ contains all terms that depend on the derivative of the currents. The electrical portion of the model (5) thus has a single multiplicative non-linearity, namely a dependence on ω_r.

In steady state, $\frac{\mathrm{d}\omega_r}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$, and equations (5) and (6) become

$$y = [\mathbf{A} + \omega_r \mathbf{B}]\, x \tag{7}$$

$$T_e(x) = T_m(\omega_r) \tag{8}$$

Matrices $\mathbf{A}$ and $\mathbf{B}$ are completely defined by the machine parameters. Since the phase angle of the applied voltage does not affect the power consumption of the motor, we specify the stator d-axis voltage V_{ds} equal to the magnitude of the bus voltage V obtained from the power flow analysis and the stator q-axis voltage V_{qs} equal to zero. The voltage is then directed entirely in the d-axis. This specification is corrected at the end of the method by rotating the current angles by the bus voltage angle δ obtained from the power flow analysis. Therefore, the voltage vector y in (7) is completely known. Since V_{qs} is specified to be zero, the active power used by the machine is $P = V_{ds}I_{ds}$. The active power is also known from the power flow analysis, so the d-axis current can be directly determined:

$$I_{ds} = \frac{P}{V_{ds}} \tag{9}$$

With predetermined values of P, Q, and V_s from a power flow model and specified machine parameters, it is not possible to satisfy all terminal conditions using only induction machine variables. In other words, with fixed values of V_s and given machine parameters, there are not sufficient degrees of freedom to specify both the active power P and reactive power Q at the machine terminals. It is common practice to enforce reactive power balance at the bus using a shunt capacitor (see [7] and [8, p. 627] for further discussion and [15, p. 963], [16, p. 20-16], and [17] for commercial software implementations using an additional capacitor). Thus, (9) does not depend on the reactive power Q.

The machine model can be put into the form of an eigenvalue problem by combining the known voltage vector and the known matrix $\mathbf{A}$. First rewrite the voltage vector y as

$$y = \begin{bmatrix} V_{ds} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{V_{ds}}{I_{ds}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} I_{ds} = \begin{bmatrix} R_A \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} I_{ds} \tag{10}$$

where, using (9), R_A is specified in terms of known terminal quantities V and P.

$$R_A = \frac{V_{ds}}{I_{ds}} = \frac{V^2}{P} \tag{11}$$

Then define the matrix $\mathbf{D}$ as

$$-\mathbf{D} = -\left[\begin{array}{c|c} R_A & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] + \mathbf{A} \tag{12}$$

and rewrite (7) as

$$-\mathbf{D}x + \omega_r \mathbf{B}x = 0 \tag{13}$$

or, in a well-known generalized eigenvalue form

$$\mathbf{D}x = \omega_r \mathbf{B}x \tag{14}$$

The formulation (14) can be solved with generalized eigenvalue solution techniques or, if $\mathbf{D}$ is invertible, converted to standard eigenvalue form and solved with standard eigenvalue techniques [25].

$$\frac{1}{\omega_r} x = \mathbf{D}^{-1}\mathbf{B}x \tag{15}$$

Since the eigenvector x can be arbitrarily scaled, rescaling the eigenvector using the known value of I_{ds} from (9) is required. Additionally, correction for the voltage angle δ from the power flow analysis is needed: rotate the current vector x by δ after solving the eigenvalue problem.

The solution to the electrical portion of the induction machine model derived in this section will match the active power input to the induction machine with the value of P obtained from the power flow model. However, the reactive power input to the induction machine will not generally match the value of reactive power Q delivered to the bus obtained from the power flow model. As discussed previously, a shunt capacitor at the machine terminals gives an additional degree of freedom that is used to enforce reactive power balance at the bus [7], [8, p. 627], [15, p. 963], [16, p. 20-16], [17]. The reactive power consumed by the induction machine is

$$Q_{machine} = V_{qs}I_{ds} - V_{ds}I_{qs} \tag{16}$$

The necessary reactive power injected by the capacitor can then be determined.

$$Q_{cap} = Q_{machine} - Q \tag{17}$$

After solving the electrical portion of the induction machine model to obtain the currents and the rotor speed ω_r, the mechanical portion of the model (2), (3) is solved to initialize the load torque model. This requires specification of the load torque model $T_m(\omega_r)$. A load torque model used in many common power system simulation packages [15], [16] is given by

$$T_m(\omega_r) = T_0 \cdot (\omega_r)^{\gamma} \tag{18}$$

where T_0 is the parameter that requires initialization and γ is a specified constant that is selected based on the type of mechanical load being driven by the induction motor. Other load torque models may be required for specific systems [16, p. 20-19]. The induction machine initial conditions model is completed by using (8) and (18) to determine values for the load torque model parameters.

We conclude this section by showing that all solutions to the induction machine initialization problem can be obtained by solving the generalized eigenvalue problem (14). The derivation in this section has made no mathematical approximations, and thus solutions to (14) are indeed solutions to the induction machine initialization problem. A finite number of eigenvalues exist for *regular* (as opposed to *singular*) generalized eigenvalue problems. Furthermore, robust codes exist for solving regular generalized eigenvalue problems [25], [26]. The generalized eigenvalue problem (14) is singular if

$$\det(\mathbf{D} - \omega_r \mathbf{B}) = 0 \tag{19}$$

for all values of ω_r (i.e. all values of ω_r are eigenvalues of (14)). Otherwise it is regular [26]. We now show that the generalized eigenvalue problem (14) is regular.

Physically meaningful choices of the machine parameters require strictly positive reactances X_s, X_m, X_{lr1}, X_{lr2} and resistances R_s, R_{r1}, R_{r2}. Although R_A has the same units as a resistance, it does not represent a physical resistance and can be either positive, when the machine is operating as a motor, or negative, when the machine is operating as a generator.

The matrix $\mathbf{B}$ is singular, and, depending on the value of R_A, the matrix $\mathbf{D}$ may be either singular or invertible. The matrix $\mathbf{D}$ is singular when $R_A = \frac{\det(\mathbf{A})}{\det(\mathbf{A}_1)}$, where the matrix $\mathbf{A}$ is defined in (5) and the matrix $\mathbf{A}_1$ is the minor formed by eliminating the first row and first column of $\mathbf{A}$.

If the matrix $\mathbf{D}$ is non-singular, invert $\mathbf{D}$ to obtain the standard eigenvalue form (15). The generalized eigenvalue problem (14) is regular for the case of non-singular $\mathbf{D}$ [26].

For the case of singular $\mathbf{D}$, the generalized eigenvalue problem (14) is regular if $\det(\mathbf{D} - \omega_r\mathbf{B}) \neq 0$ for some value of ω_r. To show this condition holds, consider a permuted version of the double-cage machine model given in (1) where the even and odd rows and columns of the matrix are gathered to emphasize the structure of this matrix. Application to other induction machine models follows from these same arguments. Define the matrix $\mathbf{G}_0$ as a function of c

$$\mathbf{G}_0(c) = \begin{bmatrix} -R_s & 0 & 0 & X_s & X_m & X_m \\ 0 & -\frac{R_{r1}}{c} & 0 & X_m & X_1 & X_m \\ 0 & 0 & -\frac{R_{r2}}{c} & X_m & X_m & X_2 \\ -X_s & -X_m & -X_m & -R_s & 0 & 0 \\ -X_m & -X_1 & -X_m & 0 & -\frac{R_{r1}}{c} & 0 \\ -X_m & -X_m & -X_2 & 0 & 0 & -\frac{R_{r2}}{c} \end{bmatrix} \tag{20}$$

where $c = 1 - \omega_r$, $X_s = X_{ls} + X_m$, $X_1 = X_{lr1} + X_m$, and $X_2 = X_{lr2} + X_m$.

From a permuted form of (19), the generalized eigenvalue problem (14) is singular if

$$\begin{aligned} &\det\left(\mathbf{U}^T(\mathbf{D} - \omega_r\mathbf{B})\mathbf{U}\right) \\ &= \det\left(\begin{bmatrix} R_A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{bmatrix} \mathbf{G}_0(c) \right) = 0 \end{aligned} \tag{21}$$

for all values of c. The permutation matrix $\mathbf{U}$ gathers the even and odd rows and columns of the matrix $(\mathbf{D} - \omega_r\mathbf{B})$ in order to emphasize the structure of this matrix, which we exploit in the remainder of this section.

Define $\mathbf{G}_1(c)$ as the minor formed by eliminating the first row and first column of $\mathbf{G}_0(c)$. Then (21) can be rewritten as

$$c^4 \det(\mathbf{G}_0(c)) + c^4 R_A \det(\mathbf{G}_1(c)) = 0 \tag{22}$$

Assuming non-zero c, (22) holds when

$$R_A = -\frac{\det(\mathbf{G}_0(c))}{\det(\mathbf{G}_1(c))} \tag{23}$$

$\mathbf{G}_0(c)$ can be rewritten as

$$\mathbf{G}_0(c) = \left[\begin{array}{c|c} -R_s & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{G}_1(c) \end{array}\right] + \left[\begin{array}{c|ccccc} 0 & 0 & 0 & X_s & X_m & X_m \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -X_s & 0 & 0 & 0 & 0 & 0 \\ -X_m & 0 & 0 & 0 & 0 & 0 \\ -X_m & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{24}$$

Since $\mathbf{G}_1(c)$ is negative definite, it is invertible. The matrix determinant lemma [27] can therefore be used to rewrite $\det(\mathbf{G}_0(c))$ as

$$\begin{aligned} \det(\mathbf{G}_0(c)) &= -R_s \det(\mathbf{G}_1(c)) \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_s & X_m & X_m \end{bmatrix} \left[\begin{array}{c|c} -\frac{1}{R_s} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{G}_1(c)^{-1} \end{array}\right] \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ X_s & 0 \\ X_m & 0 \\ X_m & 0 \end{bmatrix} \right) \\ &= -R_s \det(\mathbf{G}_1(c)) \det\left(\begin{bmatrix} 1 & \frac{1}{R_s} \\ x^T \mathbf{G}_1(c)^{-1} x & 1 \end{bmatrix} \right) \end{aligned} \tag{25}$$

where $x = \begin{bmatrix} 0 & 0 & X_s & X_m & X_m \end{bmatrix}^T$.

Simplifying and substituting (25) into (23) gives

$$R_A = R_s - x^T \mathbf{G}_1(c)^{-1} x \tag{26}$$

Since $x^T \mathbf{G}_1(c) x$ is a function of c, $x^T \mathbf{G}_1(c)^{-1} x$ is also a function of c. Thus, (26) shows that the value of R_A required for the generalized eigenvalue problem (14) to be singular is a function of c. Since R_A is only dependent on the input power P and terminal voltage magnitude V as shown in (11), R_A is a fixed value for a given induction machine initialization problem. Therefore, the singularity condition (19) does not hold for all values of ω_r and the generalized eigenvalue problem (14) is regular.

Regularity implies that (14) has a finite number of eigenvalues. Robust codes exist for solving regular generalized eigenvalue problems [25], [26]. For physically meaningful machine parameters, the matrix $\mathbf{B}$ has a two-dimensional nullspace. The generalized eigenvalue problem (14) will therefore have at least two infinite valued eigenvalues. These infinite valued eigenvalues are not physically meaningful and can be neglected. The finite eigenvalues and associated eigenvectors correspond to the physically meaningful solutions of the induction machine initialization model. Since the derivation in this section has made no mathematical approximations, solving the generalized eigenvalue problem (14) will provide all solutions to the induction machine initial conditions model.

Additionally, assuming $\mathbf{D}$ is non-singular, $\mathbf{D}^{-1}\mathbf{B}$ in (15) has at most rank four for the double-cage machine model. Therefore, the largest possible factor in the eigenvalue characteristic equation for the double-cage induction machine model is a fourth-order polynomial. Hence, the eigenvalue formulation can be non-iteratively solved using the equation for roots of quartic polynomials as given in [28]. This may be convenient for codes in large-scale simulation programs.

C. Double-Cage Induction Machine Numeric Example

Assume V_{in} is obtained from a power flow analysis. Specify $V_{ds} = V_{in}$ and $V_{qs} = 0$. The matrices $\mathbf{B}$ and $\mathbf{D}$ are

$$\mathbf{B} = \frac{1}{\omega_s}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_m & 0 & (X_{lr1}+X_m) & 0 & X_m \\ -X_m & 0 & -(X_{lr1}+X_m) & 0 & -X_m & 0 \\ 0 & X_m & 0 & X_m & 0 & (X_{lr2}+X_m) \\ -X_m & 0 & -X_m & 0 & -(X_{lr2}+X_m) & 0 \end{bmatrix} \tag{27}$$

$$\mathbf{D} = \begin{bmatrix} (R_A-R_s) & X_s & 0 & X_m & 0 & X_m \\ -X_s & -R_s & -X_m & 0 & -X_m & 0 \\ 0 & X_m & -R_{r1} & (X_{lr1}+X_m) & 0 & X_m \\ -X_m & 0 & -(X_{rl1}+X_m) & -R_{r1} & -X_m & 0 \\ 0 & X_m & 0 & X_m & -R_{r2} & (X_{lr2}+X_m) \\ -X_m & 0 & -X_m & 0 & -(X_{r2}+X_m) & -R_{r2} \end{bmatrix} \tag{28}$$

Now ω_r and the currents in x can be found by solving (14). Note that the eigenvector x must be scaled correctly. Since the first entry of x should be I_{ds}, scaling is done by multiplying each entry in the eigenvector x by I_{ds} divided by the first entry of x.

Consider a double-cage induction machine with the parameter values in per unit representation given in Table I. These parameter values were obtained from the 37 kW double-cage machine in Table 2 of [22]. Assume that the per unit values in Table II are obtained from a power flow analysis. An example power flow model that provides these values will be solved in Section III-C.

Also consider a constant torque load model representing aggregated compressor load. γ in (18) is 0 for this load torque model, and thus $T_m = T_0$.

X_{ls}	X_{lr1}	X_{lr2}	X_m
0.0572	0.1479	0.0572	1.9860
R_s	R_{r1}	R_{r2}	ω_s (rad/sec)
0.0050	0.0123	0.1129	377.0

TABLE I
DOUBLE-CAGE INDUCTION MACHINE EQUIVALENT CIRCUIT PARAMETERS

P	V_{in}	δ
2.5	1.0	$-20°$

TABLE II
POWER FLOW PARAMETERS

From this data, (9) shows that $I_{ds} = 2.500$ and (11) shows that $R_A = 0.400$. $\mathbf{D}$ can then be obtained from (28).

$$\mathbf{D} = \begin{bmatrix} 0.395 & 2.043 & 0 & 1.986 & 0 & 1.986 \\ -2.043 & -0.005 & -1.986 & 0 & -1.986 & 0 \\ 0 & 1.986 & -0.012 & 2.134 & 0 & 1.986 \\ -1.986 & 0 & -2.134 & -0.012 & -1.986 & 0 \\ 0 & 1.986 & 0 & 1.986 & -0.113 & 2.043 \\ -1.986 & 0 & -1.986 & 0 & -2.043 & -0.113 \end{bmatrix}$$

Solving (14) gives four finite solutions for ω_r (the two infinite eigenvalues are artifacts of the generalized eigenvalue specification). The eigenvector x was scaled such that the entry corresponding to I_{ds} is equal to its known value of 2.500. After scaling the eigenvector, the currents were rotated by δ. This was accomplished by calculating the vectors $I_k = (I_{dk} + jI_{qk})\,\mathrm{e}^{j\delta}$ for $k = \{s, r1, r2\}$. The rotated d-axis currents correspond to the real parts of I_k, and the rotated q-axis currents correspond to the imaginary parts of I_k. Finally, torque was calculated using (3).

The solutions for ω_r in radians per second and the currents and torque in per unit are presented in Table III. Note that four solutions are acquired rather than the single solution that would be obtained from an iterative method. Directly obtaining all solutions is an advantage of this method over traditional iterative methods. Solutions two and four are stable, while solutions one and three are unstable. The high-speed, stable solution four is presented in bold. Stability was determined by calculating the eigenvalues of the Jacobian of the linearization of the induction machine equations (1), (2), (3) [29]. All eigenvalues for stable solutions had negative real parts, whereas at least one of the eigenvalues for each unstable solution had a positive real part.

Solution	1	2	3	**4**
$\omega_r\left(\frac{\text{rad}}{\text{sec}}\right)$	-486.2	187.7	331.5	**360.3**
T_m	2.062	2.295	2.375	**2.445**
I_{ds}	-0.736	0.333	0.869	**1.610**
I_{qs}	-9.332	-6.395	-4.922	**-2.886**
I_{dr1}	1.610	1.542	-0.381	**-1.524**
I_{qr1}	2.828	4.125	4.158	**2.328**
I_{dr2}	-1.001	-2.040	-0.673	**-0.298**
I_{qr2}	6.298	1.982	0.434	**0.172**

TABLE III
SOLUTION TO DOUBLE-CAGE INDUCTION MACHINE EXAMPLE

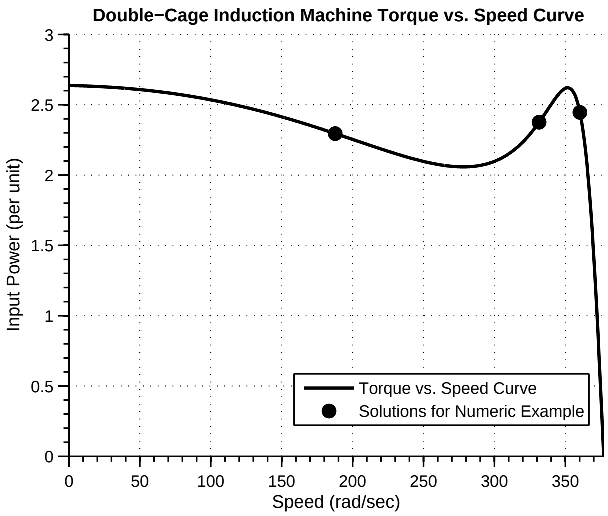

Fig. 2a. Torque vs. Speed Curve and Solutions

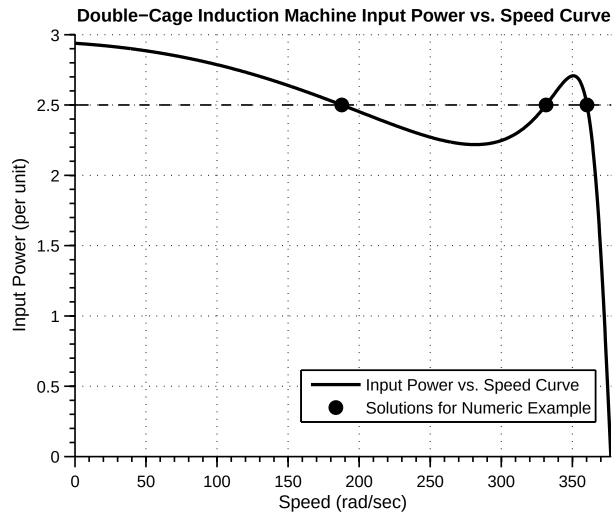

Fig. 2b. Input Power vs. Speed Curve and Solutions

The torque versus speed curve for this machine is given in Fig. 2a. Torque-speed curves with shapes similar to the curves in this figure, with increasing torque near zero speed, are common (see, for instance, the NEMA Class C torque-speed curves in references [23, p. 346] and [30, p. 337], and the discussion in [31, pp. 287-289]). Solution 2 in Table III is in this portion of the torque-speed curve. This solution is not a mathematical artifact; it is a stable operating point for the induction machine. To show the existence of such solutions, consider a dynamic simulation of the induction machine initialized from the high-speed, stable solution 4 performed using PowerWorld [17] as shown in Fig. 3. One second into the simulation, a balanced three-phase fault is applied for 0.25 seconds at the machine terminals. Note that the machine does not return to the high-voltage, stable solution 4 after the fault clears. Rather, the machine operates at a stable, lower-speed solution. (This solution is not the same as solution 2 in Table III since the reactive power demand at the machine terminals for solution 2 does not match the value of reactive power supplied from the initialization using solution 4.)

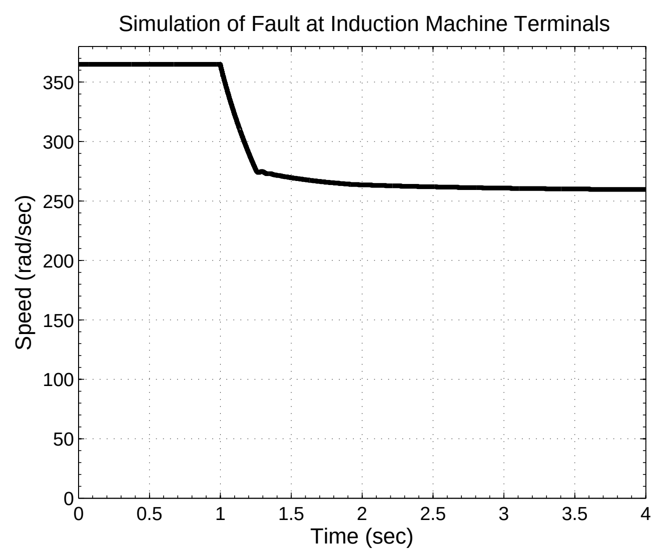

Fig. 3. Simulation of Fault at Induction Machine Terminals

Changing the power flow parameters may reduce the number of real solutions. For instance, increasing V_{in} from 1.0 to 1.10 per unit will result in two real and two complex solutions. Since the machine speed must be real-valued, the complex solutions are not physically meaningful.

While power engineers are accustomed to induction machine torque vs. speed curves, this example also benefits from investigating the input power vs. speed curve given in Fig. 2b. It is clear from Fig. 2a that the input power for the solutions is equal to the specified value of 2.50 per unit. Resisitive losses comprise an increasing proportion of the input power as the speed decreases. In fact, with positive torque and negative speed, resistive losses in solution one require both mechanical and electrical input power. This unstable, negative-speed solution may not be physical; data are not collected in [22] for the negative portion of the torque-speed curve.

Note that the solutions in Table III were obtained by neglecting core losses and magnetic saturation. A linear resistor representing core losses could be incorporated into this formulation, but this representation may not be accurate for a wide range of solutions. More detailed representations of core losses and magnetic saturation could be handled using an iterative approach: solve the model assuming linearity and use the solution to update the machine parameters, accounting for core losses and magnetic saturation. Iteration is typically not necessary for system-wide studies where errors introduced by neglecting saturation and core losses are negligible compared to uncertainties in other parameters, such as those in the load torque models.

III. A Multiparameter Eigenvalue Formulation of the Power Flow Model

A. Introduction to the Power Flow Model

The power flow model relates the active and reactive power injected at each bus to the voltage magnitude and angle at each bus. There are four variables associated with each bus i: the net active power injection (P_i), the net reactive power injection (Q_i), the voltage magnitude (V_i) and the voltage angle (δ_i).

While many derivations of the power flow model use the voltage magnitude V_i and voltage angle δ_i directly, we decompose the voltages at each bus into orthogonal d and q components. Decomposing bus voltages into orthogonal components is often done when convenient [32]–[34].

$$V_{di} = V_i \cos(\delta_i) \tag{29}$$

$$V_{qi} = V_i \sin(\delta_i) \tag{30}$$

The power flow model can be defined as

$$P_i = V_{di} \sum_{k=1}^{n} (G_{ik} V_{dk} - B_{ik} V_{qk}) + V_{qi} \sum_{k=1}^{n} (B_{ik} V_{dk} + G_{ik} V_{qk}) \tag{31}$$

$$Q_i = V_{di} \sum_{k=1}^{n} (-B_{ik} V_{dk} - G_{ik} V_{qk}) + V_{qi} \sum_{k=1}^{n} (G_{ik} V_{dk} - B_{ik} V_{qk}) \tag{32}$$

$$V_i^2 = V_{di}^2 + V_{qi}^2 \tag{33}$$

where $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ is the admittance matrix relating the voltages and currents.

While (31), (32), and (33) must all be satisfied at all buses, only two equations are directly enforced at each bus when solving the power flow model. There are three bus types in the power flow model: PQ, PV, and slack. PQ buses, which typically correspond to loads, enforce the active and reactive power equations (31) and (32). The resulting values of V_{di} and V_{qi} are used to determine the voltage magnitude V_i from (33). PV buses, which typically correspond to generators, enforce the active power and voltage magnitude equations (31) and (33). The resulting values of V_{di} and V_{qi} are used to determine the reactive power Q_i using (32). Finally, a single slack bus is specified to provide active and reactive power balance. The slack bus has specified values of V_{di} and V_{qi}. The active power P_i, reactive power Q_i, and voltage magnitude V_i at the slack bus are determined from (31), (32), and (33).

B. The Power Flow Equations Formulated as a Multiparameter Eigenvalue Problem

The k-parameter eigenvalue problem combines k eigenvalues and k equations into a single problem. The right k-parameter eigenvalue problem can be represented as in (34), where λ_j is a scalar eigenvalue, $\mathbf{M}_{ij}$ is a matrix, and x_i is an eigenvector.

$$\left(\mathbf{M}_{i0} + \sum_{j=1}^{k} \lambda_j \mathbf{M}_{ij} \right) x_i = 0, \quad i = 1, \ldots, k \tag{34}$$

The power flow equations are next described in a multiparameter eigenvalue form. Each bus contributes an additional two voltage parameters. Equations (35), (36) and (37) are the power flow equations presented in multiparameter eigenvalue

$$\begin{aligned} \Bigg(& \begin{bmatrix} 0 & 0 & -P_i \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{di} \begin{bmatrix} G_{ii} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + V_{qi} \begin{bmatrix} 0 & G_{ii} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ & + \sum_{k=1,\ldots,n \; k\neq i} \left\{ V_{dk} \begin{bmatrix} G_{ik} & B_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + V_{qk} \begin{bmatrix} -B_{ik} & G_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \Bigg) \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{35}$$

$$\begin{aligned} \Bigg(& \begin{bmatrix} 0 & 0 & -Q_i \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{di} \begin{bmatrix} -B_{ii} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + V_{qi} \begin{bmatrix} 0 & -B_{ii} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ & + \sum_{k=1,\ldots,n \; k\neq i} \left\{ V_{dk} \begin{bmatrix} -B_{ik} & G_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + V_{qk} \begin{bmatrix} -G_{ik} & -B_{ik} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \Bigg) \begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{36}$$

$$\left(\begin{bmatrix} 0 & 0 & -V_i^2 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + V_{di} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + V_{qi} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} w_{1i} \\ w_{2i} \\ w_{3i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

form for bus i and correspond to (31), (32), and (33), respectively.

The eigenvalues of the multiparameter eigenvalue formulation can be easily recognized as the V_d and V_q voltages. Expressions for the eigenvectors can be obtained by expanding the second and third rows of the corresponding multiparameter eigenvalue equation and examining the relationships between the elements of the eigenvector. For instance, the second and third rows of the multiparameter eigenvalue formulation of the active power equation (35) are

$$-V_{di}x_{2i} + V_{qi}x_{1i} = 0 \;\Rightarrow x_{2i} = \frac{V_{qi}}{V_{di}}x_{1i} \tag{38}$$

$$-x_{1i} + V_{di}x_{3i} = 0 \;\Rightarrow x_{3i} = \frac{1}{V_{di}}x_{1i} \tag{39}$$

Thus, the eigenvector x_i can be rewritten as

$$x_i = \begin{bmatrix} x_{1i} \\ \frac{V_{qi}}{V_{di}}x_{1i} \\ \frac{1}{V_{di}}x_{1i} \end{bmatrix} = x_{1i}\frac{1}{V_{di}}\begin{bmatrix} V_{di} \\ V_{qi} \\ 1 \end{bmatrix} \tag{40}$$

Since eigenvectors have a single degree of freedom in their magnitude (if v is an eigenvector, then av is also an eigenvector for scalar $a \neq 0$), assuming $V_{di} \neq 0$ and $x_{1i} \neq 0$, (40) can be rewritten as

$$x_i = \begin{bmatrix} V_{di} \\ V_{qi} \\ 1 \end{bmatrix} \tag{41}$$

Since the second and third rows of all right multiparameter equations are identical, (35), (36), and (37) have identical eigenvectors

$$x_i = y_i = w_i = \begin{bmatrix} V_{di} \\ V_{qi} \\ 1 \end{bmatrix} \tag{42}$$

C. Direct Solution for Two Bus Systems

The theory of multiparameter eigenvalue problems is not as mature as the theory of other eigenvalue problems. Much of multiparameter eigenvalue theory assumes that the $\mathbf{M}_{ij}$ matrices are Hermitian or that the matrix $\left(\mathbf{M}_{i0} + \sum_{j=1}^{k} \lambda_j \mathbf{M}_{ij}\right)$ is left or right definite. For instance, the books [35] and [36] work almost entirely with Hermitian multiparameter eigenvalue problems. The multiparameter eigenvalue formulation of the power flow equations is neither Hermitian nor left or right definite and thus cannot be analyzed with theory developed for specialized forms of multiparameter eigenvalue problems.

However, multiparameter eigenvalue theory does enable the general solution of two-parameter eigenvalue problems. Since each bus besides the slack bus has two degrees of freedom, the number of parameters necessary to represent an n-bus power system is $2(n-1)$. Hence, two-bus systems with specified slack bus voltage can be represented as a two-parameter eigenvalue problem.

The power flow model for two-bus systems can be easily solved using many existing solution techniques [8]. In fact, solutions to two-bus systems can be expressed explicitly. Solving the power flow model for two-bus systems with the multiparameter eigenvalue formulation has no numerical advantages over existing techniques. This section is intended to demonstrate the validity of applying multiparameter eigenvalue solution techniques to the power flow model. Future advances in multiparameter eigenvalue theory may enable application to larger systems or give other insights into the power flow model. This is discussed further in Section III-D.

Since the matrices used in the power flow equations (35), (36), and (37) are small (3×3), the Kronecker product method described in Chapter 2 of [37] can be used to solve the power flow model for two-bus systems. This method converts a two-parameter eigenvalue problem into a set of generalized eigenvalue problems. Consider the two-parameter eigenvalue problem described in (43) and (44), where λ_1 and λ_2 are eigenvalues and x_1 and x_2 are eigenvectors.

$$\mathbf{A}_0 x_1 + \lambda_1 \mathbf{A}_1 x_1 + \lambda_2 \mathbf{A}_2 x_1 = 0 \tag{43}$$

$$\mathbf{B}_0 x_2 + \lambda_1 \mathbf{B}_1 x_2 + \lambda_2 \mathbf{B}_2 x_2 = 0 \tag{44}$$

(43) and (44) can be rewritten using the Kronecker product method as two generalized eigenvalue problems.

$$\mathbf{\Delta}_1 z = \lambda_1 \mathbf{\Delta}_0 z \tag{45}$$

$$\mathbf{\Delta}_2 z = \lambda_2 \mathbf{\Delta}_0 z \tag{46}$$

where

$$\mathbf{\Delta}_0 = \mathbf{A}_1 \otimes \mathbf{B}_2 - \mathbf{A}_2 \otimes \mathbf{B}_1 \tag{47}$$

$$\mathbf{\Delta}_1 = \mathbf{A}_2 \otimes \mathbf{B}_0 - \mathbf{A}_0 \otimes \mathbf{B}_2 \tag{48}$$

$$\mathbf{\Delta}_2 = \mathbf{A}_0 \otimes \mathbf{B}_1 - \mathbf{A}_1 \otimes \mathbf{B}_0 \tag{49}$$

$$z = x_1 \otimes x_2 \tag{50}$$

The Kronecker product is denoted by $\otimes$. See [28] for further discussion on the Kronecker product. The solution to the two-parameter eigenvalue problem can be obtained from a simultaneous solution of (45) and (46). Applying this to a two-bus power system, where bus 1 is the slack bus with known values of V_{d1} and V_{q1}, bus 2 is a PQ bus with active power injection P_2 and reactive power injection Q_2, and the system has a 2×2 admittance matrix $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$, gives the following result.

$$\mathbf{\Delta}_1 z = V_{d2} \mathbf{\Delta}_0 z \tag{51}$$

$$\mathbf{\Delta}_2 z = V_{q2} \mathbf{\Delta}_0 z \tag{52}$$

where

$$\mathbf{\Delta}_{0,\ PQ} = \begin{bmatrix} 0 & -G_{22}B_{22} & 0 & G_{22}B_{22} & 0 & 0 & 0 & 0 & 0 \\ G_{22} & 0 & 0 & 0 & G_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -G_{22} & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 & B_{22} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (53)$$

$$\mathbf{\Delta}_{1,\ PQ} = \begin{bmatrix} 0 & aB_{22} & 0 & -bG_{22} & (aG_{22}+bB_{22}) & -Q_2G_{22} & 0 & -P_2B_{22} & 0 \\ -a & 0 & 0 & -b & 0 & 0 & P_2 & 0 & 0 \\ 0 & 0 & 0 & -G_{22} & 0 & 0 & 0 & 0 & 0 \\ -b & a & -Q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -B_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (54)$$

$$\mathbf{\Delta}_{2,\ PQ} = \begin{bmatrix} (-aB_{22}+bG_{22}) & -aG_{22} & Q_2G_{22} & -bB_{22} & 0 & 0 & P_2B_{22} & 0 & 0 \\ 0 & -a & 0 & 0 & -b & 0 & 0 & P_2 & 0 \\ G_{22} & 0 & a & 0 & 0 & b & 0 & 0 & -P_2 \\ 0 & 0 & 0 & -b & a & -Q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ B_{22} & 0 & 0 & 0 & 0 & 0 & b & -a & Q_2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (55)$$

and, for notational convenience, $a = G_{21}V_{d1} - B_{21}V_{q1}$ and $b = B_{21}V_{d1} + G_{21}V_{q1}$.

Solving (51) and (52) provides the possible values for both V_{d2} and V_{q2}, respectively. To determine which pairs of eigenvalues are actually solutions to the power flow equations, test each of the possible voltage combinations (V_{d2}, V_{q2}) in the power flow equations (31) and (32). Voltage combinations that yield the specified values of active and reactive power injections P_i and Q_i are solutions to the power flow equations.

A similar process can be done for a two-bus system where bus 2 is a PV bus with active power injection P_2 and voltage magnitude V_2. (51) and (52) must be solved, where $\mathbf{\Delta}_{0,\ PV}$, $\mathbf{\Delta}_{1,\ PV}$, and $\mathbf{\Delta}_{2,\ PV}$ are given in the appendix. Each combination of eigenvalues must be checked in the power flow equations for active power (31) and voltage magnitude (33). Voltage combinations that satisfy these equations are the solutions to the power flow model.

Direct solutions for systems with more than two buses requires general solution methods for multiparameter eigenvalue problems with more than two parameters. No solution techniques for this category of problems were found in any existing literature. Therefore, new developments in multiparameter eigenvalue theory will be needed before direct solution of the multiparmeter eigenvalue formulation of the power flow model will be possible for practical size power systems.

We next present a numeric example of the two-bus system shown in Fig. 4. This example completes the induction machine initialization example given in Section II-C. Bus 1 is a slack bus with known values of V_{d1} and V_{q1}, and bus 2 is a PQ bus with known values of power injections P_2 and Q_2, as given in Table IV. The capacitor at bus 2 is used to balance the difference in reactive power between the induction machine and power flow models [7].

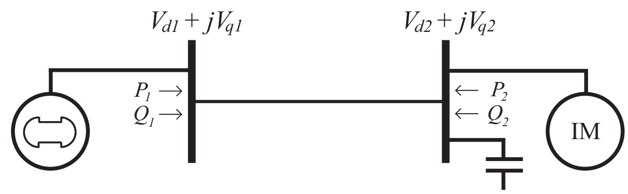

Fig. 4. Two Bus System

V_{d1}	V_{q1}	P_2	Q_2
1.05	0	-2.500	0.180

TABLE IV
TWO BUS SYSTEM SPECIFIED POWER FLOW VALUES

The transmission line is assumed to have impedance $0.0050 + j0.1433$ per unit, or equivalently, admittance $0.2432 - j6.9699$ per unit. Thus, the system has an admittance matrix given by

$$\mathbf{Y} = \mathbf{G} + j\mathbf{B} = \begin{bmatrix} 0.2432 - j6.9699 & -0.2432 + j6.9699 \\ -0.2432 + j6.9699 & 0.2432 - j6.9699 \end{bmatrix}$$

Using these values to solve (51) indicates that for any solution to the power flow equations, V_{d2} must take one of the following values: $\{0, 0.1103, 0.9397\}$. Similarly, solving (52) indicates that V_{q2} must take one of the following values: $\{-14.4413, -0.3420, 30.4857\}$.

The two solutions to the two-bus power flow model can be obtained by testing each of the 9 possible combinations of V_{d2} and V_{q2} in the power flow equations (31) and (32). The two combinations of voltage components that satisfy the power flow equations are $V_{d2} + jV_{q2} = 0.9397 - j0.3420$ and $V_{d2} + jV_{q2} = 0.1103 - j0.3420$.

The high voltage solution $V_{d2} + jV_{q2} = 0.9397 - j0.3420$ is equivalent to the $1.0 \angle -20°$ per unit voltage specified in the double-cage induction machine example in section II-C. The input power of 2.5 per unit at bus 2 is also identical to the induction machine example. Thus, the dynamic power system simulation initialization problem can be completed for all devices connected to bus 2 by specifying the reactive power supplied by the capacitor. Assuming the induction machine is operating at the high-speed, stable solution 4 in Table III, the reactive power consumption of the machine is determined from

(16) to be 2.161 per unit. From (17), the capacitor must supply the difference between the value of reactive power injection Q_2 required by the the power flow model, -1.401 per unit, and the reactive power consumed by the induction machine: $Q_{cap} = 2.161 - (-0.180) = 2.342$ per unit reactive power.

D. Discussion

The multiparameter eigenvalue formulation of the power flow model enables future advances in eigenvalue theory to benefit power systems research. Further developments in direct solution techniques for multiparameter eigenvalue problems may enable the determination of all solutions to the power flow model for power systems with more than two buses. However, it is important to note that much of the existing multiparameter eigenvalue research has focused on limited problems, such as restrictions to Hermitian matrices or definiteness requirements. The matrices resulting from the power flow equations do not have these properties. Therefore, future developments in direct solution techniques for multiparameter eigenvalue problems must not be dependent on special matrix properties in order to be applicable to the power flow model.

The multiparameter eigenvalue formulation of the power flow equations may still prove useful even if relevant direct solution techniques are not developed. For instance, discovery of an upper bound to the number of real solutions of a multiparameter eigenvalue problem would be useful for the continuation power flow method [11]. Given an initial solution, the continuation power flow finds multiple solutions to the power flow equations by tracing from one solution to the next. The tracing method varies a single parameter, such as the active power injection at a bus, until a new solution is found. The continuation power flow may continue tracing after actually finding all solutions. An upper bound on the number of real solutions provided by future advances in multiparameter eigenvalue theory would offer the continuation power flow a stop condition to prevent unnecessary tracing.

The future discovery of conditions for the existence of real solutions to multiparameter eigenvalue problems is an additional advancement in multiparameter eigenvalue theory that has the potential for practical application to the power flow model. The power flow model under some operating conditions, particularly when the system is heavily loaded, may not have any real-valued solutions (since the power flow equations in rectangular coordinates are derived by separating the real and imaginary parts of the complex power injections, physically meaningful solutions for the voltage components must be real). On the other hand, solutions to solvable systems may not be found due to convergence problems inherent to numeric solution codes. If a solution is not found, engineers may be uncertain of whether there truly are no solutions or if the numeric code is not converging to a solution that does in fact exist. Conditions for the existence of real solutions to multiparameter eigenvalue problems would eliminate this uncertainty. Also, for some purposes, such as determining the locations of bifurcations, the details of a solution may not be particularly important; it may be sufficient to simply determine whether any real solutions exist for a given power flow model. With future research into conditions for the existence of real solutions, the multiparameter eigenvalue formulation of the power flow model could play a role for these purposes as well.

IV. Conclusion

The induction machine initial conditions model has been reformulated as a generalized eigenvalue problem that can be solved using standard linear algebra techniques. The rotor speed can be obtained from the eigenvalue and the stator and rotor currents can be obtained from the eigenvector. The torque is a function of the currents and can therefore be directly determined after solving the eigenvalue formulation. In contrast to traditional iterative methods, the eigenvalue method has the advantage of providing all solutions to the initial conditions model and can reliability determine when no solutions exist. We found this tremendously useful for certain research problems in which knowledge of the multiple solutions is important [19]. This paper has developed the eigenvalue formulation for a double-cage induction machine model and has given a numeric example.

The power flow model was formulated as a multiparameter eigenvalue problem. After introducing this formulation, expressions for the eigenvectors were derived. The two-parameter eigenvalue formulation can be solved directly using the Kronecker product method for two-bus systems with both PV and PQ buses. This method converts the two-parameter eigenvalue formulation into a set of generalized eigenvalue problems that must be simultaneously satisfied.

The multiparameter eigenvalue formulation of the power flow model enables advances in multiparameter eigenvalue theory to contribute to power system engineering knowledge. Specific advances that would be beneficial include the development of general solution techniques for multiparameter eigenvalue problems with more than two parameters, a method for determining the number of real solutions, and conditions for the existence of any real solutions to multiparameter eigenvalue problems.

Appendix

The $\mathbf{\Delta}_{0,\,PV}$, $\mathbf{\Delta}_{1,\,PV}$, and $\mathbf{\Delta}_{2,\,PV}$ matrices for the two-bus system with a PV and a slack bus are given in (56), (57), and (58).

$$\mathbf{\Delta}_{0,\,PV} = \begin{bmatrix} 0 & G_{22} & 0 & -G_{22} & 0 & 0 & 0 & 0 & 0 \\ G_{22} & 0 & 0 & 0 & G_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -G_{22} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{56}$$

$$\mathbf{\Delta}_{1,\,PV} = \begin{bmatrix} 0 & -a & 0 & 0 & -b & -V_2^2 G_{22} & 0 & P_2 & 0 \\ -a & 0 & 0 & -b & 0 & 0 & P_2 & 0 & 0 \\ 0 & 0 & 0 & -G_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -V_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{57}$$

$$\mathbf{\Delta}_{2,\,PV} = \begin{bmatrix} a & 0 & V_2^2 G_{22} & b & 0 & 0 & -P_2 & 0 & 0 \\ 0 & -a & 0 & 0 & -b & 0 & 0 & P_2 & 0 \\ G_{22} & 0 & a & 0 & 0 & b & 0 & 0 & -P_2 \\ 0 & 0 & 0 & 0 & 0 & -V_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_2^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{58}$$

For notational convenience, $a = G_{21}V_{d1} - B_{21}V_{q1}$ and $b = B_{21}V_{d1} + G_{21}V_{q1}$.

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