A Sufficient Condition for Global Optimality of Solutions to the Optimal Power Flow Problem

Daniel K. Molzahn, *Student Member, IEEE*, Bernard C. Lesieutre, *Senior Member, IEEE*, and Christopher L. DeMarco, *Member, IEEE*

***Abstract*—Recent applications of a semidefinite programming (SDP) relaxation to the optimal power flow (OPF) problem offers a polynomial time method to compute a global optimum for a large subclass of OPF problems. In contrast, prior OPF solution methods in the literature guarantee only local optimality for the solution produced. However, solvers employing SDP relaxation remain significantly slower than mature OPF solution codes. This letter seeks to combine the advantages of the two methods. In particular, we develop a SDP-inspired sufficient condition test for global optimality of a candidate OPF solution. This test may then be easily applied to a candidate solution generated by a traditional, only-guaranteed-locally-optimal OPF solver.**

***Index Terms*—Optimal power flow, Global optimization**

I. Introduction

The optimal power flow (OPF) problem determines an optimal operating point for an electric power system in terms of a specified objective function, subject to both network equality constraints (i.e., the power flow equations, which model the relationship between voltages and power injections) and engineering limits (e.g., inequality constraints on voltage magnitudes, active and reactive power generations, and flows on transmission lines and transformers).

Recent research has applied semidefinite programming (SDP) to the OPF problem [1]. Using a rank relaxation, the OPF problem is formulated as a convex SDP. If the relaxed problem satisfies a rank condition, a global optimum of the OPF problem can be determined in polynomial time. No prior OPF solution method guarantees calculation of the global solution in polynomial time; SDP thus has a substantial advantage over other solution techniques. However, the rank condition is not always satisfied, so the SDP relaxation does not give physically meaningful solutions to all OPF problems [2].

The SDP relaxation of the OPF problem is computationally limited by a positive semidefinite constraint on a $2n \times 2n$ matrix, where n is the number of buses in the system. Thus, despite being provably polynomial time, the SDP relaxation is computationally challenging for large systems. With recent work in matrix completion decompositions that speed computation by exploiting power system sparsity, solution of the SDP relaxation is feasible for large systems [3], [4].

However, solution of the SDP relaxation is still significantly slower than mature OPF algorithms, such as interior point methods [5]. It would be beneficial to pair the solution speed of mature OPF solution algorithms with the global optimality guarantee of the SDP relaxation. We propose a sufficient condition derived from the Karush-Kuhn-Tucker (KKT) conditions for optimality of the SDP relaxation of the OPF problem [6]. A candidate solution obtained from a mature OPF solution algorithm that satisfies the KKT conditions is guaranteed to be globally optimal. However, satisfaction of these conditions is not necessary for global optimality.

University of Wisconsin-Madison Department of Electrical and Computer Engineering: molzahn@wisc.edu, lesieutre@engr.wisc.edu, demarco@engr.wisc.edu. This work was supported by the Applied Mathematics activity, Advanced Scientific Computing Research program in the DOE Office of Science under contract from Argonne National Laboratory.

II. Sufficient Condition for Global Optimality

Consider an n-bus power system, where $\mathcal{N}$ is the set of all buses, $\mathcal{G}$ is the set of generator buses, and $\mathcal{L}$ is the set of all lines. $P_{Dk} + jQ_{Dk}$ is the load demand and $V_k = V_{dk} + jV_{qk}$ is the voltage phasor at buses $k \in \mathcal{N}$. $P_{Gk} + jQ_{Gk}$ is the generation at buses $k \in \mathcal{G}$. S_{lm} is the apparent power flow on the line $(l, m) \in \mathcal{L}$. Lines are modeled as Π-equivalent circuits (see [4] for more flexible models). Superscripts "max" and "min" denote upper and lower limits. $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ is the network admittance matrix. Define a quadratic objective function associated with each generator $k \in \mathcal{G}$, typically representing a variable operating cost. The OPF problem is

$$\min \sum_{k\in\mathcal{G}} \left(c_{k2}P_{Gk}^2 + c_{k1}P_{Gk} + c_{k0}\right) \qquad \text{subject to} \tag{1a}$$

$$P_{Gk}^{\min} \leq P_{Gk} \leq P_{Gk}^{\max} \qquad \forall k \in \mathcal{G} \tag{1b}$$

$$Q_{Gk}^{\min} \leq Q_{Gk} \leq Q_{Gk}^{\max} \qquad \forall k \in \mathcal{G} \tag{1c}$$

$$\left(V_k^{\min}\right)^2 \leq V_{dk}^2 + V_{qk}^2 \leq \left(V_k^{\max}\right)^2 \qquad \forall k \in \mathcal{N} \tag{1d}$$

$$|S_{lm}| \leq S_{lm}^{\max} \qquad \forall (l,m) \in \mathcal{L} \tag{1e}$$

$$P_{Gk} - P_{Dk} = V_{dk}\sum_{i=1}^{n}(\mathbf{G}_{ik}V_{di} - \mathbf{B}_{ik}V_{qi}) + V_{qk}\sum_{i=1}^{n}(\mathbf{B}_{ik}V_{di} + \mathbf{G}_{ik}V_{qi}) \tag{1f}$$

$$Q_{Gk} - Q_{Dk} = V_{qk}\sum_{i=1}^{n}(\mathbf{G}_{ik}V_{di} - \mathbf{B}_{ik}V_{qi}) - V_{dk}\sum_{i=1}^{n}(\mathbf{B}_{ik}V_{di} + \mathbf{G}_{ik}V_{qi}) \tag{1g}$$

A solution to (1) consists of vectors of voltage phasors $V = V_d + jV_q$, power injections $P + jQ$, and Lagrange multipliers. We denote the Lagrange multipliers associated with the voltage magnitude equation (1d) as μ, those associated with the active power balance equation (1f) as λ, those associated with the reactive power balance equation (1g) as γ, and those associated with the apparent power line flow equation (1e) as ζ.

In order to reformulate the standard OPF problem into a structure that allows it to be solved as a SDP, it is necessary to define several matrices that embed the network's bus admittance matrix information into larger arrays. Following the development of [1], let e_k be the k^{th} standard basis vector in $\mathbb{R}^n$. Define the matrices $Y_k = e_k e_k^T \mathbf{Y}$ and $Y_{lm} = \left(\frac{jb_{lm}}{2} + y_{lm}\right) e_l e_l^T - (y_{lm})\, e_l e_m^T$, where b_{lm} is the line's shunt susceptance, y_{lm} is the line's series admittance, and superscript T indicates the transpose operator. Define matrices

$$\mathbf{Y}_k = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\left(Y_k + Y_k^T\right) & \mathrm{Im}\left(Y_k^T - Y_k\right) \\ \mathrm{Im}\left(Y_k - Y_k^T\right) & \mathrm{Re}\left(Y_k + Y_k^T\right) \end{bmatrix} \tag{2a}$$

$$\bar{\mathbf{Y}}_k = -\frac{1}{2}\begin{bmatrix} \mathrm{Im}\left(Y_k + Y_k^T\right) & \mathrm{Re}\left(Y_k - Y_k^T\right) \\ \mathrm{Re}\left(Y_k^T - Y_k\right) & \mathrm{Im}\left(Y_k + Y_k^T\right) \end{bmatrix} \tag{2b}$$

$$\mathbf{M}_k = \begin{bmatrix} e_k e_k^T & \mathbf{0} \\ \mathbf{0} & e_k e_k^T \end{bmatrix} \tag{2c}$$

$$\mathbf{Y}_{lm} = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\left(Y_{lm} + Y_{lm}^T\right) & \mathrm{Im}\left(Y_{lm}^T - Y_{lm}\right) \\ \mathrm{Im}\left(Y_{lm} - Y_{lm}^T\right) & \mathrm{Re}\left(Y_{lm} + Y_{lm}^T\right) \end{bmatrix} \tag{2d}$$

$$\bar{\mathbf{Y}}_{lm} = -\frac{1}{2}\begin{bmatrix} \mathrm{Im}\left(Y_{lm} + Y_{lm}^T\right) & \mathrm{Re}\left(Y_{lm} - Y_{lm}^T\right) \\ \mathrm{Re}\left(Y_{lm}^T - Y_{lm}\right) & \mathrm{Im}\left(Y_{lm} + Y_{lm}^T\right) \end{bmatrix} \tag{2e}$$

Define the matrix variable $\mathbf{W} = xx^T$ where $x = \begin{bmatrix} V_{d1} & \cdots & V_{dn} & V_{q1} & \cdots & V_{qn} \end{bmatrix}^T$. Formulate OPF problem (1) in terms of $\mathbf{W}$ as in [1]: bus k active and reactive power injections are $\mathrm{trace}\left(\mathbf{Y}_k \mathbf{W}\right)$ and $\mathrm{trace}\left(\bar{\mathbf{Y}}_k \mathbf{W}\right)$ and squared voltage magnitude is $\mathrm{trace}\left(\mathbf{M}_k \mathbf{W}\right)$; active and reactive flows on line (l,m) are $\mathrm{trace}\left(\mathbf{Y}_{lm} \mathbf{W}\right)$ and $\mathrm{trace}\left(\bar{\mathbf{Y}}_{lm} \mathbf{W}\right)$. The SDP relaxation is formed by replacing the constraint $\mathbf{W} = xx^T$ with $\mathbf{W} \succeq 0$, where $\succeq 0$ indicates positive semidefiniteness.

The $\mathbf{A}$ matrix of the dual SDP problem, which collects terms of the optimality conditions as in [1], requires Lagrange multipliers in terms of the square of voltage magnitudes (denoted as ξ) rather than the voltage magnitudes themselves. Use the chain rule of differentiation for the conversion

$$\xi_k = \mu_k \left(\frac{1}{2V_{k_0}}\right) \tag{3}$$

where V_{k_0} is the solution's voltage magnitude at bus k. Additionally, the solution to (1) gives line-flow limit Lagrange multipliers ζ in terms of apparent power (MVA), but the dual SDP problem requires separate multipliers in terms of active and reactive power flows (denoted as α and β, respectively). Using the relationship $S_{lm} = \sqrt{P_{lm}^2 + Q_{lm}^2}$, where P_{lm} and Q_{lm} are the active and reactive flows, respectively, on the line from bus l to bus m, the appropriate conversions are

$$\alpha_{lm} = \zeta_{lm}\left(\frac{\partial S_{lm}}{\partial P_{lm}}\right) = \zeta_{lm}\left(\frac{P_{lm_0}}{S_{lm_0}}\right) \tag{4a}$$

$$\beta_{lm} = \zeta_{lm}\left(\frac{\partial S_{lm}}{\partial Q_{lm}}\right) = \zeta_{lm}\left(\frac{Q_{lm_0}}{S_{lm_0}}\right) \tag{4b}$$

where P_{lm_0} and Q_{lm_0} are the solution's flows on line lm. The $\mathbf{A}$ matrix is then

$$\mathbf{A} = \sum_{k\in\mathcal{N}} \left(\lambda_k \mathbf{Y}_k + \gamma_k \bar{\mathbf{Y}}_k + \xi_k \mathbf{M}_k\right) + \sum_{(l,m)\in\mathcal{L}} \left(\alpha_{lm} \mathbf{Y}_{lm} + \beta_{lm} \bar{\mathbf{Y}}_{lm}\right) \tag{5}$$

When feasible, the SDP relaxation has a global solution that satisfies the KKT conditions for optimality [6]. A candidate OPF solution may satisfy these KKT conditions, in which case the solution is globally optimal. Using $\mathbf{W} = xx^T$ and $\mathbf{A}$ from (5), the first KKT condition of complementarity is

$$\mathrm{trace}\left(\mathbf{AW}\right) = 0 \tag{6}$$

The second regards feasibility of the $\mathbf{W}$ and $\mathbf{A}$ matrices. These matrices are feasible in the SDP relaxation if they are positive semidefinite. The matrix $\mathbf{W} = xx^T$ is positive semidefinite by construction. Thus, the relevant feasibility condition is

$$\mathbf{A} \succeq 0 \tag{7}$$

III. Discussion

Satisfaction of both (6) and (7) implies global optimality regardless of the rank characteristics of the $\mathbf{A}$ matrix (i.e., $\dim\left(\mathrm{null}\left(\mathbf{A}\right)\right) \leq 2$ is not required). Non-zero branch resistances, as necessary in [1], are not required. However, enforcing small minimum branch resistances may result in satisfaction of (6) and (7) for problems that would not otherwise satisfy these conditions.

If either (6) or (7) is not satisfied, global optimality is indeterminate. Failure to satisfy these conditions may result when the semidefinite relaxation does not satisfy the rank condition [2], in which case the solution may still be globally optimal but is not guaranteed to be so. Alternatively, failure to satisfy (6) and (7) may indicate that a better solution exists.

When applied to the IEEE test systems [7] without minimum resistances, global optimality of solutions from MATPOWER's interior point algorithm [5] was verified for the 14, 30, and 57-bus systems, but not for the 118 and 300-bus systems due to non-satisfaction of (7). With a minimum branch resistance of 1×10^{-4} per unit, the solution to the 118-bus system (but not the 300-bus system) was verified to be globally optimal. Note that tight solution tolerances are often needed to obtain satisfactory numerical results.

IV. Conclusion

Using the KKT conditions of a semidefinite relaxation of the OPF problem, this letter has proposed a sufficient condition test for global optimality of a candidate OPF solution.

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