Investigation of Non-Zero Duality Gap Solutions to a Semidefinite Relaxation of the Optimal Power Flow Problem

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Abstract

Recently, a semidefinite programming relaxation of the power flow equations has been applied to the optimal power flow problem. When this relaxation is "tight" (i.e., the solution has zero duality gap), a globally optimal solution is obtained. Existing literature investigates sufficient conditions whose satisfaction guarantees zero duality gap solutions. However, there is limited study of non-zero duality gap solutions. By illustrating the feasible spaces for optimal power flow problems and their semidefinite relaxations, this paper investigates examples of non-zero duality gap solutions. Results for large system models suggest that non-convexities associated with small subsections of the network are responsible for non-zero duality gap solutions.

1. Introduction

The optimal power flow (OPF) problem seeks decision variable values to yield an optimal operating point for an electric power system in terms of a specified objective and subject to engineering inequality constraints (e.g., active and reactive power generation, bus voltage magnitudes, line flows, etc.) and network equality constraints (i.e., the power flow equations). Total generation cost is the typical objective; other objectives, such as loss minimization, may be considered.

The OPF problem is non-convex due to the non-linear power flow equations [1] and is, in general, NP-hard [2]. Non-convexity of the OPF problem has made solution techniques an ongoing research topic since the problem was first introduced by Carpentier in 1962 [3]. Many OPF solution techniques have been proposed; see [4]–[9] for relevant survey papers. Traditional solution methods use iterative techniques that are dependent on an initial guess and only guaranteed to obtain locally optimal solutions.

Recent advances in optimization techniques provide new means for addressing power systems problems. Specifically, a semidefinite relaxation of the power flow equations has been applied to the OPF problem [2], [10]. Applying convex optimization techniques using a semidefinite relaxation allows for global solution of many OPF problems in polynomial time. When the relaxation is "tight" (i.e., satisfies a rank condition for obtaining a zero duality gap solution), a globally optimal solution is recoverable.

Although the semidefinite relaxation yields zero duality gap solutions (i.e., the relaxation is "tight") for many OPF problems, there are practical OPF problems which have non-zero duality gap solutions [11]–[13]. Such solutions provide lower bounds on the optimal objective value but do not give physically meaningful solutions to the original engineering quantities in the OPF problem. Existing literature studies cases for which the semidefinite relaxation of the OPF problem is tight by providing sufficient conditions for zero duality gap solutions. These conditions include highly limiting requirements on power injection and voltage magnitude limits and either radial networks (typical only of distribution system models) or unrealistically dense placement of controllable phase shifting transformers [14]–[17]. Research explaining why the semidefinite relaxation of the OPF problem may yield solutions with non-zero duality gap is limited to [12] and [13], which present test OPF problems with locally optimal solutions in the feasible space.

Applications of semidefinite programming to the OPF and other power systems problems would benefit from understanding the causes of non-zero duality gap solutions. One potential cause of a non-zero duality gap solution is a disconnected feasible space with components near a global optimum. This cause of non-zero duality gap solutions can be considered using the geometry of the feasible space of the semidefinite relaxation. The semidefinite relaxation forms a convex space that contains the entire feasible space defined by the power flow equations. Nearby disconnected components may result in the semidefinite relaxation finding a solution "between" the disconnected components of the feasible space defined by the power flow equations which is nonetheless in feasible space of the semidefinite program. This paper expands on the two-bus test system from [12] and provides an additional three-bus example system with disconnected feasible space that yields a non-zero duality gap solution. Existing work in this area includes an archive of test cases with local optima [13]. For these systems, application of the semidefinite relaxation of the OPF problem has limited success in obtaining zero duality gap solutions; eight of the ten test cases with local optima yield non-zero duality gap solutions for some choice of parameters.

Non-zero duality gap solutions may also result from other types of non-convexity inherent to the power flow equations. The semidefinite relaxation of the OPF problem yields a non-zero duality gap solution to a five-bus example from [1], which has connected but non-convex feasible space.

Using insights from these small systems, we study larger systems that yield non-zero duality gap solutions. Using the rank one matrix closest to the non-zero duality gap solution, we evaluate the active and reactive power "mismatches" to the injections specified at load buses. For the cases studied, this analysis shows that small subsets of the network have large mismatches while the mismatches at the majority of the buses are insignificant. For some systems with non-zero duality gap solutions, we find minor perturbations in specified system data which result in zero duality gap solutions. In other words, we find that small problematic subsets of the network may cause non-zero duality gap solutions. Perturbations to these subsets of the network resulted in zero duality gap solutions. However, these perturbations were determined heuristically by examining the power injection mismatches and could only be determined for some systems; no robust method of identifying such modifications has yet been identified.

Further analysis shows that radially connecting a small system with non-zero duality gap solution to a larger system with zero duality gap solution results in the solution to the merged system having non-zero duality gap. This also suggests that non-zero duality gap solutions to large system models may be due to non-convexity in a small subset of the system.

This paper is organized as follows. Section 2 presents the OPF problem. Section 3 discusses non-zero duality gap solutions to the semidefinite relaxation of the OPF problem, including an exploration of the relevant feasible spaces. Section 4 gives concluding comments and future research directions. An extended description of this work is available in Chapter 7 of [18], which additionally studies non-zero duality gap solutions to semidefinite formulations for determining multiple solutions to the power flow equations [11] and for determining voltage stability margins [19], [20].

2. The Optimal Power Flow Problem

There has been significant interest in applying semidefinite programming to the OPF problem. The OPF problem adds engineering limits on voltage magnitudes, generator active and reactive power injections, line flows, etc. to the physical constraints of the power flow equations. A typical objective is to minimize generation cost, which is specified as a convex quadratic function of generator active power injections. In this section, we present the OPF problem in both the classical and semidefinite relaxation formulations.

2.1. The Classical Formulation of the OPF Problem

Consider an n-bus power system, where $\mathcal{N} = \{1, 2, \ldots, \mathrm{n}\}$ represents the set of all buses, $\mathcal{G}$ represents the set of generator buses, and $\mathcal{L}$ represents the set of all lines. Let $P_{Dk} + jQ_{Dk}$ represent the active and reactive load demand at each bus $k \in \mathcal{N}$. Define $V_{dk} + jV_{qk}$ as the voltage phasor at bus $k \in \mathcal{N}$. Let P_{Gk} and Q_{Gk} represent the active and reactive outputs of generator $k \in \mathcal{G}$. Superscripts "max" and "min" denote specified upper and lower limits. Let $S_{lm}^{\max}$ represent the maximum apparent power flow allowed on line $(l, m) \in \mathcal{L}$. Let $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ denote the network admittance matrix. $f_k(P_{Gk}) = c_{k2}P_{Gk}^2 + c_{k1}P_{Gk} + c_{k0}$ is the convex quadratic cost function for generator $k \in \mathcal{G}$.

The classical formulation of the OPF problem is

$$\min_{P_G,\, Q_G,\, S,\, V_d,\, V_q} \sum_{k \in \mathcal{G}} f_k\left(P_{Gk}\right) \quad \text{subject to} \tag{1a}$$

$$P_{Gk}^{\min} \le P_{Gk} \le P_{Gk}^{\max} \qquad \forall k \in \mathcal{G} \tag{1b}$$

$$Q_{Gk}^{\min} \le Q_{Gk} \le Q_{Gk}^{\max} \qquad \forall k \in \mathcal{G} \tag{1c}$$

$$\left(V_k^{\min}\right)^2 \le V_{dk}^2 + V_{qk}^2 \le \left(V_k^{\max}\right)^2 \qquad \forall k \in \mathcal{N} \tag{1d}$$

$$\left|S_{lm}\right| \le S_{lm}^{\max} \qquad \forall (l, m) \in \mathcal{L} \tag{1e}$$

$$P_{Gk} - P_{Dk} = V_{dk}\sum_{i=1}^{n}\left(\mathbf{G}_{ki}V_{di} - \mathbf{B}_{ki}V_{qi}\right) + V_{qk}\sum_{i=1}^{n}\left(\mathbf{B}_{ki}V_{di} + \mathbf{G}_{ki}V_{qi}\right) \qquad \forall k \in \mathcal{N} \tag{1f}$$

$$Q_{Gk} - Q_{Dk} = V_{dk}\sum_{i=1}^{n}\left(-\mathbf{B}_{ki}V_{di} - \mathbf{G}_{ki}V_{qi}\right) + V_{qk}\sum_{i=1}^{n}\left(\mathbf{G}_{ki}V_{di} - \mathbf{B}_{ki}V_{qi}\right) \qquad \forall k \in \mathcal{N} \tag{1g}$$

Constraints (1b)–(1e) enforce engineering limits on active and reactive power injection, voltage magnitudes, and apparent-power line flows. Constraints (1f) and (1g) are the power flow equations associated with the transmission network.

2.2. A Semidefinite Relaxation of the OPF Problem

We next present a semidefinite relaxation of the OPF problem adopted from [2]. First define the vector of voltage coordinates

$$x = \begin{bmatrix} V_{d1} & V_{d2} & \cdots & V_{dn} & V_{q1} & V_{q21} & \cdots & V_{qn} \end{bmatrix}^T \quad (2)$$

Then define the rank one matrix

$$\mathbf{W} = xx^T \quad (3)$$

Let e_i denote the i^{th} standard basis vector in $\mathfrak{R}^n$. Define the matrix $Y_i = e_i e_i^T \mathbf{Y}$. Constant matrices employed in the semidefinite relaxation are then

$$\mathbf{Y}_i = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\left(Y_i + Y_i^T\right) & \mathrm{Im}\left(Y_i^T - Y_i\right) \\ \mathrm{Im}\left(Y_i - Y_i^T\right) & \mathrm{Re}\left(Y_i + Y_i^T\right) \end{bmatrix} \quad (4)$$

$$\overline{\mathbf{Y}}_i = -\frac{1}{2}\begin{bmatrix} \mathrm{Im}\left(Y_i + Y_i^T\right) & \mathrm{Re}\left(Y_i - Y_i^T\right) \\ \mathrm{Re}\left(Y_i^T - Y_i\right) & \mathrm{Im}\left(Y_i + Y_i^T\right) \end{bmatrix} \quad (5)$$

$$\mathbf{M}_i = \begin{bmatrix} e_i e_i^T & 0 \\ 0 & e_i e_i^T \end{bmatrix} \quad (6)$$

The expression $\mathrm{tr}(\mathbf{Y}_i \mathbf{W})$, where tr is the matrix trace operator, gives the active power injection at bus *i*. The expressions $\mathrm{tr}(\overline{\mathbf{Y}}_i \mathbf{W})$ and $\mathrm{tr}(\mathbf{M}_i \mathbf{W})$ give the reactive power injection and squared voltage magnitude, respectively, at bus *i*.

Constant matrices are also needed to represent the active and reactive line flows on Π model lines. Define the matrix $Y_{lm} = \left(\frac{b_{lm}}{2} + y_{lm}\right) e_l e_m{}^T$, where b_{lm} is the total shunt susceptance and y_{lm} is the series admittance of the line. The constant matrices are

$$\mathbf{Y}_{lm} = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\left(Y_{lm} + Y_{lm}^T\right) & \mathrm{Im}\left(Y_{lm}^T - Y_{lm}\right) \\ \mathrm{Im}\left(Y_{lm} - Y_{lm}^T\right) & \mathrm{Re}\left(Y_{lm} + Y_{lm}^T\right) \end{bmatrix} \quad (7)$$

$$\overline{\mathbf{Y}}_{lm} = -\frac{1}{2}\begin{bmatrix} \mathrm{Im}\left(Y_{lm} + Y_{lm}^T\right) & \mathrm{Re}\left(Y_{lm} - Y_{lm}^T\right) \\ \mathrm{Re}\left(Y_{lm}^T - Y_{lm}\right) & \mathrm{Im}\left(Y_{lm} + Y_{lm}^T\right) \end{bmatrix} \quad (8)$$

The expressions $\mathrm{tr}(\mathbf{Y}_{lm}\mathbf{W})$ and $\mathrm{tr}(\overline{\mathbf{Y}}_{lm}\mathbf{W})$ give the active and reactive power flow from bus *l* to bus *m*, respectively.

Replacement of the rank one constraint (3) by the less stringent constraint $\mathbf{W} \succeq 0$, where $\succeq$ indicates the corresponding matrix is positive semidefinite, yields the semidefinite relaxation. The semidefinite relaxation is "tight" (i.e., has zero duality gap) if the **W** matrix of a globally optimal solution has rank one. The semidefinite relaxation of the OPF problem is then

$$\min_{\mathbf{W}} \sum_{k \in \mathcal{G}} \alpha_k \quad \text{subject to} \quad (9a)$$

$$P_{Gk}^{\min} - P_{Dk} \le \mathrm{tr}(\mathbf{Y}_k \mathbf{W}) \le P_{Gk}^{\max} - P_{Dk} \quad \forall k \in \mathcal{G} \quad (9b)$$

$$Q_{Gk}^{\min} - Q_{Dk} \le \mathrm{tr}(\overline{\mathbf{Y}}_k \mathbf{W}) \le Q_{Gk}^{\max} - Q_{Dk} \quad \forall k \in \mathcal{G} \quad (9c)$$

$$\left(V_k^{\min}\right)^2 \le \mathrm{tr}(\mathbf{M}_k \mathbf{W}) \le \left(V_k^{\max}\right)^2 \quad \forall k \in \mathcal{N} \quad (9d)$$

$$\begin{bmatrix} -\left(S_{lm}^{\max}\right)^2 & \mathrm{tr}(\mathbf{Y}_{lm}\mathbf{W}) & \mathrm{tr}(\overline{\mathbf{Y}}_{lm}\mathbf{W}) \\ \mathrm{tr}(\mathbf{Y}_{lm}\mathbf{W}) & -1 & 0 \\ \mathrm{tr}(\overline{\mathbf{Y}}_{lm}\mathbf{W}) & 0 & -1 \end{bmatrix} \preceq 0 \quad \forall (l,m) \in \mathcal{L} \quad (9e)$$

$$\begin{bmatrix} \alpha_k - c_{k1}\Gamma(\mathbf{W}) - c_{k0} & -\sqrt{c_{k2}}\,\Gamma(\mathbf{W}) \\ -\sqrt{c_{k2}}\,\Gamma(\mathbf{W}) & 1 \end{bmatrix} \succeq 0 \quad \forall k \in \mathcal{G} \quad (9f)$$

$$\mathbf{W} \succeq 0 \quad (9g)$$

where for notational purposes $\Gamma(\mathbf{W}) = \mathrm{tr}(\mathbf{Y}_k \mathbf{W}) + P_{Dk}$.

This formulation allows no more than a single generator per bus and does not allow parallel lines. More flexible modeling is described in [21]. A solution to (9) has zero duality gap if **W** satisfies the rank condition

$$\mathrm{rank}(\mathbf{W}) \le 2 \quad (10)$$

and a matrix defined by the dual variables of the semidefinite optimization problem (9) has a two-dimensional nullspace (see Theorem 2 and Corollary 1 in [2].)

If a solution to (9) satisfies the rank condition (10) and has dual variables that define a matrix satisfying the two-dimensional condition, a rank one matrix can be obtained by specifying the reference angle, which then allows for extracting a globally optimal voltage profile [2]. A solution to (9) that does not satisfy these conditions has non-zero duality gap. Such solutions lower bound the objective function value but do not solve the classical OPF problem (1).

3. Non-Zero Duality Gap Solutions to the OPF Problem

3.1. Feasible Space Exploration

The semidefinite relaxation does not yield zero duality gap solutions for all practical OPF problems [11]–[13]. A solution to (9) with non-zero duality gap provides a lower bound on the optimal objective value of the OPF problem, but does not yield a physically meaningful solution (i.e., a non-zero duality gap solution does not provide a voltage profile that satisfies the power flow equations (1f) and (1g)). One explanation for non-zero duality gap solutions is non-convexity due to a disconnected feasible space. This source of non-convexity is first explored using the two-bus example system from [12], which is reproduced as Figure 1.

The line in this system has impedance $R + jX = 0.04 + j0.20$ per unit and does not have a line-flow limit. The load demand at bus 2 is $P_{D2} + jQ_{D2} = 3.525 - j3.580$ per unit using a 100 MVA base. There are no limits on active and reactive power injections at bus 1. The voltage magnitude at bus 1 is constrained to the range $[0.95, 1.05]$ per unit. The voltage magnitude at bus 2 has a lower bound of 0.95 and an upper bound of $V_2^{\max}$. We use an objective of minimizing the cost of a \$1/MWh active power generation at bus 1.

Using bus 1 as the angle reference, $V_{q1} = 0$. We first consider the case where the upper voltage magnitude limit at bus 2, $V_2^{\max}$, is 1.05 per unit. In Figure 2, the entire feasible space for the non-relaxed problem (i.e., the squares of the three non-zero voltage components V_{d1}^2, V_{d2}^2, and V_{q2}^2) is plotted as the red line with two disconnected segments. The semidefinite relaxation has six degrees of freedom corresponding to the entries in the upper triangle of the **W** matrix. The conic shape in Figure 2 results from projecting this six-dimensional feasible space into three dimensions. The colors of the conic shape represent the objective value for each point in the space of the semidefinite relaxation.

Figure 2 shows that both the semidefinite relaxation and the non-relaxed feasible spaces share a global minimum, which is marked with a square in the figure. Consequentially, the semidefinite relaxation has a zero duality gap solution. (The optimal objective value is \$444.08 per hour.)

Next consider the case where $V_2^{\max} = 1.02$ per unit. This limit is illustrated by the gray plane cutting through Figure 2. This tighter limit reduces the feasible space to the region that is to the right of this plane. We now see that the global minimum in the space of the semidefinite relaxation (circle with objective value \$449.82 per hour) does not match the minimum of the non-relaxed problem (triangle with objective value \$456.55 per hour). Accordingly, the solution to the semidefinite relaxation has non-zero duality gap.

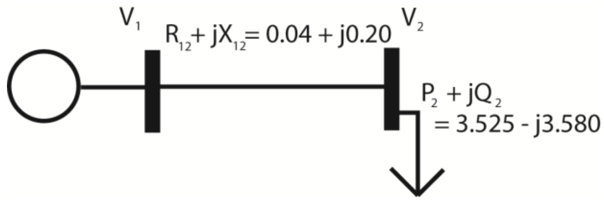

Figure 1. Two-bus system from [12]

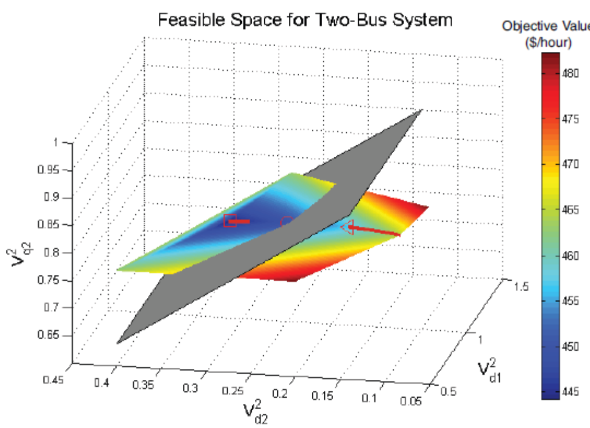

Figure 2. Feasible space for two-bus system

This example illustrates how non-zero duality gap solutions result when the non-relaxed space has components that are nearby but disconnected from the component of the feasible space containing the global optimum. The semidefinite relaxation of the OPF problem finds a solution that is not in the feasible space of the non-relaxed problem but is in the feasible space of the semidefinite program. That is, the semidefinite relaxation has an optimal solution "between" the disconnected components of the feasible space defined by the power flow equations.

A three-bus system adopted from [11] provides another example of a case where the semidefinite relaxation has a non-zero duality gap solution due to a disconnected feasible space. Figure 3 shows the diagram for this system. Bus 1 has an active power load of 1.0 per unit using a 100 MVA base. The generators at buses 1 and 2 are constrained to inject positive active power, but have no other limits on active or reactive power generation. The generator at bus 3 is a synchronous condenser which outputs zero active power and has no limit on reactive power output. The line parameters are given in Table 1. The line connecting buses 2 and 3 has an apparent-power line-flow limit of 1.0 per unit. This example uses cost functions of \$3/MWh for active power generation at bus 1 and \$1/MWh for active power generation at bus 2.

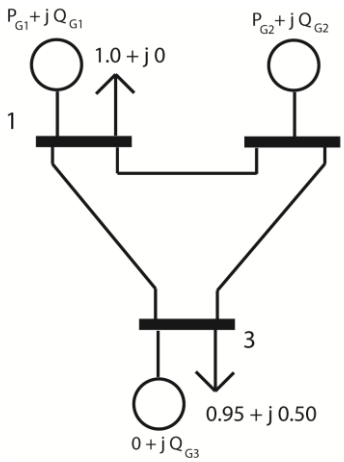

Figure 3. Three-bus system

Table 1. Line parameters for three-bus system (per unit)

From Bus	To Bus	Resistance	Reactance	Shunt Susceptance
1	3	0.065	0.62	0.45
2	3	0.025	0.75	0.70
1	2	0.042	0.90	0.30

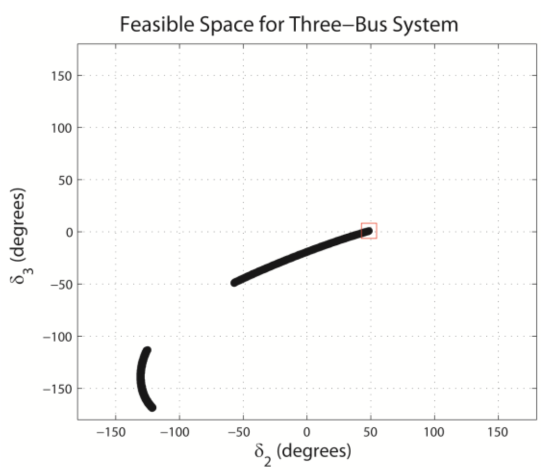

Figure 4. Feasible space for three-bus system

Voltage magnitudes at each bus are fixed to 1.0 per unit. With fixed voltage magnitudes and bus 1 providing an angle reference, this system has two degrees of freedom in the voltage angles at buses 2 and 3 (δ_2 and δ_3), which are related to the rectangular voltage components as $\tan\delta_2 = V_{q2}/V_{d2}$ and $\tan\delta_3 = V_{q3}/V_{d3}$. In Figure 4, the feasible space for the non-relaxed problem is visualized in a two-dimensional space of the voltage angles δ_2 and δ_3. The optimal solution to the non-relaxed problem, which is obtained using exhaustive search of the feasible space, has objective value of \$235.19 per hour and is marked with a square in Figure 4. The space of voltage angles used for Figure 4 does not allow for easily representing the feasible space of the semidefinite relaxation.

With an apparent-power line-flow limit of 1.0 per unit for the line between buses 2 and 3, the semidefinite relaxation yields a solution with objective value of \$234.62 per hour, which is 0.24% smaller than the result obtained using exhaustive search. A non-zero duality gap solution results from the non-convexity associated with the disconnected feasible space evident in Figure 4.

Note that OPF problems with a disconnected feasible space may still have zero duality gap solutions. That is, a disconnected feasible space is not a sufficient condition for obtaining a non-zero duality gap solution. For instance, a less stringent but still binding apparent-power line-flow limit of 1.05 per unit between buses 2 and 3 yields a disconnected feasible space with a zero duality gap solution.

In addition to a disconnected feasible space, other sources of non-convexity may result in non-zero duality gap solutions to the semidefinite relaxation of the OPF problem. This is next illustrated with a five-bus example system from [1] (reproduced as Figure 5), which has a connected but non-convex feasible space. All buses in this system are constrained to have 1.0 per unit voltage magnitude. All line flows are unconstrained, and the line reactances are specified in Figure 5. (The system is lossless since all line resistances are set to zero.) The generators at buses 1 and 2 have non-negative active power generation, and the generators at buses 3, 4, and 5 are synchronous condensers with zero active power generation. There are no limits on reactive power injection for any generator. The load demand at bus 3 is allowed to be any non-negative value $P_{D3} \geq 0$. Equations describing the feasible space for the corresponding OPF problem in terms of the voltage angles are given in [1].

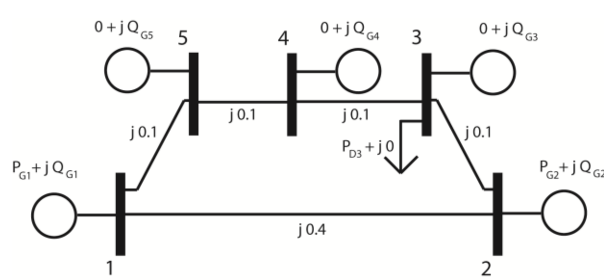

Figure 5. Five-bus system from [1]

Since the network is lossless, the system-wide active power balance imposes the equality

$$P_{D3} = P_{G1} + P_{G2} \qquad (11)$$

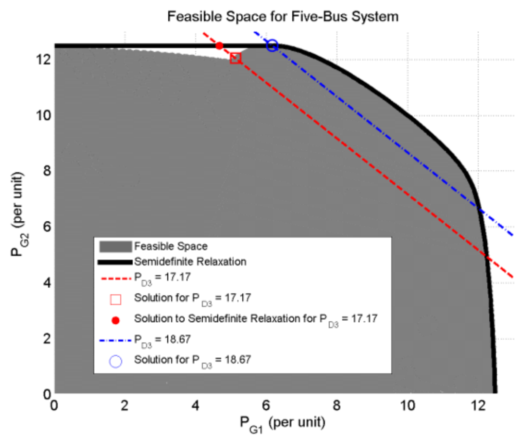

Figure 6. Feasible space for five-bus system

Figure 6 shows the feasible space of active power injections in the P_{G2} vs. P_{G1} plane. Observe that the feasible space is connected but non-convex. Projecting the semidefinite relaxation into this space yields the region enclosed by the solid black line in Figure 6. The semidefinite relaxation is tight when viewed in this space for the boundary points on the right side of the feasible space.

For this problem, the semidefinite relaxation appears to form the convex hull of the injection region in Figure 6 (i.e., the black line appears to form the convex hull of the non-convex gray region). Note, however, that this is not always the case; the semidefinite relaxation does not always yield the convex hull of the original, non-convex feasible set of power injections.

To illustrate a case where the non-convexity results in a non-zero duality gap solution, consider a load demand $P_{D3} = 17.17$ per unit. All combinations of P_{G1} and P_{G2} that satisfy the equality (11) resulting from this load demand are on the red dashed line in Figure 6. Consider the case where the generator at bus 1 is more expensive than the generator at bus 2 such that the optimal solution occurs when P_{G1} is minimized.

The globally optimal solution to the OPF problem is located at the red square in Figure 6, while the solution to the semidefinite relaxation is located at the red dot. Thus, the semidefinite relaxation yields a non-zero duality gap solution for these values of P_{D3} and generator costs. Even though the feasible space with specified P_{D3}, which consists of the one-dimensional intersection between the red dashed line from (11) and the gray region in Figure 6, is connected and convex, the non-convexity of the gray region in Figure 6 still results in a non-zero duality gap solution to the semidefinite relaxation. Note that non-zero duality gap solutions still occur when the lines have small resistances (e.g., 1×10^{-3} per unit).

3.2. Non-Zero Duality Gap Solutions to Large OPF Problems

With increased understanding of how non-convexity affects the tightness of the semidefinite relaxation for small OPF problems, we next study non-zero duality gap solutions to larger OPF problems. Solving the semidefinite relaxation for large-scale OPF problems requires exploitation of power system sparsity [21], [22]. As reported in [21], some large test cases exhibit non-zero duality gap solutions.

First proposed in [21], one metric for the duality gap is based on the mismatch between the calculated and specified active and reactive power injections at load buses. To recover a candidate voltage profile, form the closest rank one matrix to the solution's **W** matrix using the eigenvector associated with the largest eigenvalue of **W**.[1] If the solution has zero duality gap, the matrix **W** is rank one and the resulting voltage profile will satisfy the power injection equality constraints at the load buses. Conversely, the closest rank one matrix to a solution with non-zero duality gap will typically not yield a voltage profile that satisfies the power injection equality constraints at load buses. Thus, the mismatch between the calculated and specified power injections at load buses provides a measure for satisfaction of the rank condition (10).

In this paper, we specifically consider non-zero duality gap solutions to the IEEE 300-bus [25] and Polish 3012-bus [8] systems. Figures 7 and 8 show the mismatch between the specified and calculated active and reactive power injections at load buses for the IEEE 300-bus and Polish 3012-bus systems, respectively, sorted in order of increasing active power mismatch. (Note that minimum resistances of 1×10^{-4} per unit are enforced in accordance with [2].) The large

[1] Despite the fact that the underlying space of rank one matrices is non-convex, eigen decomposition yields the closest rank one matrix to the higher rank **W** matrix as measured using the Frobenius norm (i.e., square root of the sum of squares of the difference in matrix entries) or any other unitarily invariant norm. If the matrix has non-repeated eigenvalues, the closest rank one matrix is unique. See, e.g., references [23] and [24]. Pairs of repeated eigenvalues in **W** are artifacts of the lack of an angle reference specification in (9). After specifying an angle reference, eigen decomposition yields the unique closest rank one matrix to **W**.

power mismatches indicate non-zero duality gap solutions for these systems.

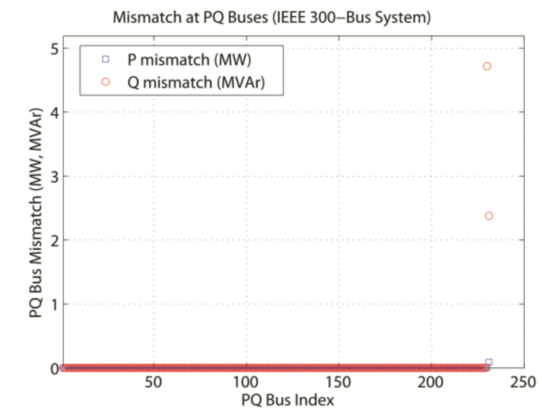

Figure 7. Power mismatch for IEEE 300-bus system

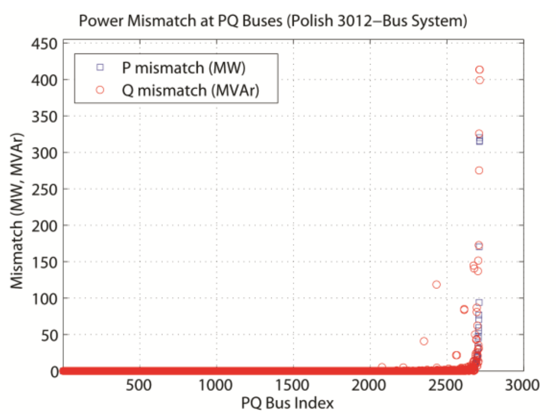

Figure 8. Power mismatch for Polish 3012-bus system

The voltage profile obtained from the closest rank one matrix to **W** yields small mismatches for the majority of buses, but a few buses display large mismatches in both active and reactive power injections. These results suggest that there are small subsections of the network that are responsible for the non-zero duality gap solutions to these systems.

To further investigate this phenomenon, we create new systems by radially connecting the two and three-bus systems shown in Figures 1 and 3 to IEEE test systems [25]. The OPF problems for the IEEE test systems have zero duality gap solutions, while, as shown in Section 3.1, non-convexities in the two and three-bus systems result in non-zero duality gap solutions. The semidefinite relaxations of the connected OPF problems have non-zero duality gap solutions. That is, non-convexities introduced in a small subset of an OPF problem may result in a non-zero duality gap solution to a problem for which the semidefinite relaxation is otherwise tight.

For example, consider a 15-bus system resulting from radial connection of bus 2 from the two-bus system in Figure 1 to bus 1 of the IEEE 14-bus system [25] using the same line impedance as in the two-bus system. If no reactive power limits are enforced for the generator at bus 1, the resulting 15-bus system has non-convexity due to a disconnected feasible space in the same manner as shown in Figure 2. Accordingly, the semidefinite relaxation has a non-zero duality gap solution. Connections of the two-bus system to other generator buses in the IEEE 14 and 30-bus systems also result in non-zero duality gap solutions. Similar test cases resulting from radial connection of the three-bus system in Figure 3 to the IEEE 14 and 30-bus systems also exhibit non-zero duality gap solutions.

These results support the conjecture that non-convexity associated with small subsets of the IEEE 300-bus and Polish 3012-bus systems are responsible for non-zero duality gap solutions. Since large systems have many opportunities to have such non-convex subsections, the semidefinite relaxations of large problems are likely to have non-zero duality gap solutions. (Limited access to large-scale system models precludes empirical evaluation of this conjecture. See [21] for duality gap analysis using the few publicly available large models.)

However, non-zero duality gap solutions with non-convexities that are limited to small regions of the network may provide close initial points for local search algorithms. Further, small perturbations to OPF problems may yield zero duality gap solutions. We next provide such perturbations for the test systems used in Section 3.1.

For the two-bus system in Figure 1, changing the line reactance from 0.20 per unit to 0.215 per unit

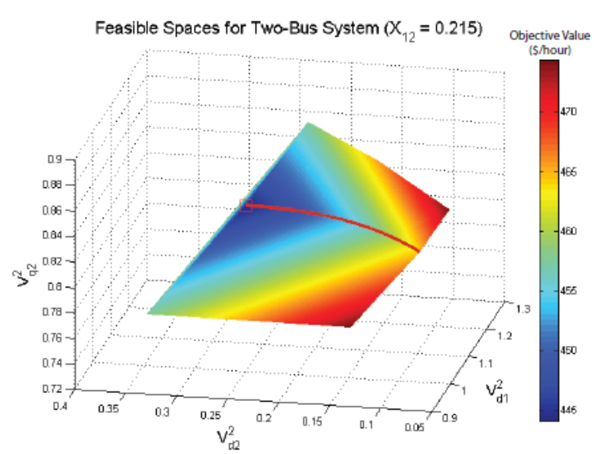

Figure 9. Feasible space for two-bus system with X_{12} = 0.215 per unit

(a 7.5% increase) results in zero duality gap solutions for any value of $V_2^{\max} \geq V_2^{\min} = 0.95$ per unit. The feasible space for the two-bus system with a line reactance of 0.215 per unit and $V_2^{\max} = 1.05$ per unit is shown in Figure 9. Since the power flow equations for this problem form a connected feasible space, any valid choice of $V_2^{\max}$ will result in the semidefinite relaxation finding a zero duality gap solution to this problem.

As mentioned previously for the three-bus system in Figure 3, replacing the apparent-power line-flow limit of 1.0 per unit with a less stringent but still binding limit of 1.05 per unit (a 5% increase) yields a zero duality gap solution.

For the five-bus system in Figure 5, changing the load demand P_{D3} from 17.17 per unit to any value greater than 18.67 per unit (an 8.7% increase) results in a zero duality gap solution. (The OPF problem is infeasible for values of P_{D3} greater than 20 per unit. The semidefinite relaxation yields non-zero duality gap solutions for any positive value of P_{D3} smaller than 18.67 per unit, which is shown using the blue dashed line in Figure 6.)

Similar perturbations may be capable of yielding tight semidefinite relaxations for large OPF problems with non-zero duality gap solutions that result from non-convexities that are isolated to small subsections of the network. Perturbations that yield zero duality gap solutions for the IEEE 300-bus system include increasing the upper bounds on voltage magnitudes at buses 23 and 7023 from 1.06 to 1.08 per unit (a 1.9% increase) and either changing the reactance of the line between buses 9533 and 9053 from 0.75 to 0.1875 per unit (a 75% decrease) or reducing the linear cost term for the generator at bus 9053 from $40/MWh to $38/MWh (a 5% decrease). (Note that, in accordance with [2], minimum resistances of 1×10^{-4} per unit are enforced on all lines.) The solutions to these perturbed systems have maximum active and reactive power mismatches less than 0.1 MW/MVAr at all load buses, which is the default Newton solution tolerance used by the commercial power flow solution package PSS/E [26].

These perturbations were obtained heuristically by iteratively changing constraint and cost parameters near buses with large mismatches (i.e., the buses corresponding to large values in Figures 8 and 9). There is no guarantee that this approach is valid for all systems. For instance, we were unable to obtain a set of perturbations that yields a zero duality gap solution for the Polish 3012-bus system.

4. Conclusion

Although the semidefinite relaxation of the optimal power flow (OPF) problem is often “tight,” practical problems may have non-zero duality gap solutions. This paper investigates non-convexities associated with non-zero duality gap solutions to the OPF problem.

Non-convexity associated with a disconnected feasible space may result in a non-zero duality gap solution. Illustrative examples are provided along with visualizations of relevant feasible spaces. OPF problems for two and three-bus systems with disconnected feasible spaces exhibit non-zero duality gap solutions.

This paper also presents a five-bus system with connected but non-convex feasible space. An OPF problem associated with this system has a non-zero duality gap solution, which demonstrates that a disconnected feasible space is not necessary for non-zero duality gap solutions.

Non-zero duality gap solutions for large OPF problems are also studied. Specifically, the IEEE 300-bus and Polish 3012-bus systems are found to exhibit non-zero duality gap solutions as evidenced by matrices that have rank greater than two. The closest rank one matrices to these non-zero duality gap solutions satisfy the power injection equations at the majority of load buses; mismatch at a small number of load buses suggests that the non-convexities causing the non-zero duality gap are isolated to a few small subsections of the network. This is supported by non-zero duality gap solutions to example cases created by radially connecting small test systems with non-zero duality gap solutions to IEEE test cases for which the semidefinite relaxation is tight. Non-convexity introduced in a small subsection of the network is sufficient to cause non-zero duality gap solutions. Further, for many cases where the semidefinite relaxation is not tight, heuristically-determined perturbations to small subsections of the network often result in problems with zero duality gap solutions.

Non-convexity that is isolated to small subsections of the network and the ability to obtain zero duality gap solutions to some perturbed OPF problems suggest directions for future research. One potential direction is development of a robust method for identifying non-convex subsections of the network before solving an OPF problem. A possible approach is to categorize common network structures that lead to non-convexity. Identification of common non-convexities may lead to future development of sufficient conditions for which satisfaction guarantees a non-zero duality gap solution to the semidefinite relaxation of the OPF problem.

Another potential research direction is development of a systematic method for determining perturbations that result in zero duality gap solutions. Ideally, such a method would find the smallest perturbations necessary in order to obtain the "closest" OPF problem for which the semidefinite relaxation is tight. Perturbations within the uncertainty associated with power system data would be particularly useful in practice. Research on this topic may draw on such works as [16], [27], and [28], which claim that zero duality gap solutions always result for OPF problems modified with a sufficient number of appropriately placed controllable phase shifting transformers along with allowing for load oversatisfaction. These modifications may be viewed as method for perturbing the original OPF problem to a nearby problem that has zero duality gap solution.

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6. References

[1] B.C. Lesieutre and I.A. Hiskens, "Convexity of the Set of Feasible Injections and Revenue Adequacy in FTR Markets," *IEEE Transactions on Power System,* vol. 20, no. 4, pp. 1790-1798, November 2005.

[2] J. Lavaei and S. Low, "Zero Duality Gap in Optimal Power Flow Problem," *IEEE Transactions on Power Systems,* vol. 27, no. 1, pp. 92-107, February 2012.

[3] J. Carpentier, "Contribution to the Economic Dispatch Problem," *Bull. Soc. Franc. Elect,* vol. 8, no. 3, pp. 431-447, 1962.

[4] M. Huneault and F. Galiana, "A Survey of the Optimal Power Flow Literature," *IEEE Transactions on Power Systems,* vol. 6, no. 2, pp. 762-70, May 1991.

[5] J. Momoh, R. Adapa, and M. El-Hawary, "A Review of Selected Optimal Power Flow Literature to 1993: I. Nonlinear and Quadratic Programming Approaches," *IEEE Transactions on Power Systems,* vol. 14, no. 1, pp. 96-104, February 1999.

[6] J. Momoh, M. El-Hawary, and R. Adapa, "A Review of Selected Optimal Power Flow Literature to 1993. II. Newton, Linear Programming and Interior Point Methods," *IEEE Transactions on Power Systems,* vol. 14, no. 1, pp. 105-111, February 1999.

[7] Z. Qiu, G. Deconinck, and R. Belmans, "A Literature Survey of Optimal Power Flow Problems in the Electricity Market Context," in *Power Systems Conference and Exposition, 2009. PSCE '09. IEEE/PES,* March 2009, pp. 1-6.

[8] R. Zimmerman, C. Murillo-Sánchez, and R. Thomas, "MATPOWER: Steady-State Operations, Planning, and Analysis Tools for Power Systems Research and Education," *IEEE Transactions on Power Systems,* vol. 26, no. 1, pp. 12-19, February 2011.

[9] H. Wang and R.J. Thomas, "Toward Reliable Computation of Large-Scale Market-Based Optimal Power Flow," *40th Annual Hawaii Conference on System Sciences, HICSS 2007*, January 2007.

[10] X. Bai, H. Wei, K. Fujisawa, and Y. Wang, "Semidefinite Programming for Optimal Power Flow Problems," *International Journal of Electrical Power & Energy* Systems, vol. 30, no. 6-7, pp.383-392, 2008.

[11] B.C. Lesieutre, D.K. Molzahn, A.R. Borden, and C.L. DeMarco, "Examining the Limits of the Application of Semidefinite Programming to Power Flow Problems," in *49th Annual Allerton Conference on Communication, Control, and Computing, 2011*, September 28-30 2011.

[12] W.A. Bukhsh, A. Grothey, K.I. McKinnon, and P.A. Trodden, "Local Solutions of Optimal Power Flow," Univrsity of Edinburgh School of Mathematics, Tech. Rep. ERGO 11-017, 2011, [Online]. Available: http://www.maths.ed.ac.uk/ERGO/pubs/ERGO-11-017.html.

[13] W.A. Bukhsh, A. Grothey, K. McKinnon, and P. Trodden, "Local Solutions of Optimal Power Flow Problem," To appear in *IEEE Transactions on Power Systems*.

[14] B. Zhang and D. Tse, "Geometry of Feasible Region of Power Networks," in *49th Annual Allerton Conference on Communication, Control, and Computing, 2011,* September 28-30 2011.

[15] S. Bose, D. Gayme, S. Low, and K. Chandy, "Optimal Power Flow Over Tree Networks," in *49th Annual Allerton Conference on Communication, Control, and Computing, 2011*, September 28-30 2011.

[16] S. Sojoudi and J. Lavaei, "Physics of Power Networks Makes Hard Optimization Problems Easy To Solve," in *2012 IEEE Power & Energy Society General Meeting,* July 22-27 2012.

[17] J. Lavaei, D. Tse, and B. Zhang, "Geometry of Power Flows in Tree Networks," in *2012 IEEE Power & Energy Society General Meeting,* July 22-27 2012.

[18] D.K. Molzahn, “Application of Semidefinite Optimization Techniques to Problems in Electric Power Systems,” *Doctoral Dissertation, University of Wisconsin–Madison Department of Electrical and Computer Engineering*, August 2013.

[19] D.K. Molzahn, B.C. Lesieutre, and C.L. DeMarco, “A Sufficient Condition for Power Flow Insolvability with Applications to Voltage Stability Margins,” *IEEE Transactions on Power Systems*, vol. 28, no. 3, pp. 2592-2601, August 2013.

[20] D.K. Molzahn, V. Dawar, B.C. Lesieutre, and C.L. DeMarco, “Sufficient Conditions for Power Flow Insolvability Considering Reactive Power Limited Generators with Applications to Voltage Stability Margins,” in *Bulk Power System Dynamics and Control - IX. Optimization, Security, and Control of the Emerging Power Grid, 2013 IREP Symposium*, August 25-30 2013.

[21] D.K. Molzahn, J.T. Holzer, B.C. Lesieutre, and C.L. DeMarco, “Implementation of a Large-Scale Optimal Power Flow Solver Based on Semidefinite Programming,” To appear in *IEEE Transactions on Power Systems*.

[22] R. Jabr, “Exploiting Sparsity in SDP Relaxations of the OPF Problem,” *IEEE Transactions on Power Systems*, vol. 27, no. 2, pp. 1138-1139, May 2012.

[23] C. Eckart and G. Young, "The Approximation of One Matrix by Another of Lower Rank," *Psychometrika*, vol. 1, no. 3, pp. 211-218, 1936.

[24] L. Mirsky, "Symmetric Gauge Functions and Unitarily Invariant Norms," *The Quarterly Journal of Mathematics*, vol. 11, no. 1, pp. 50-59, 1960.

[25] Power Systems Test Case Archive. University of Washington Department of Electrical Engineering. [Online]. Available: http://www.ee.washington.edu/research/pstca/

[26] Siemens PTI, “Volume II: Program Application Guide,” *Power System Simulation for Engineering (PSS/E)*, version 31.0, December 2007.

[27] M. Farivar and S. Low, “Branch Flow Model: Relaxations and Convexification–Part I,” *IEEE Transactions on Power Systems*, vol. 28, no. 3, pp. 2554-2564, August 2013.

[28] M. Farivar and S. Low, “Branch Flow Model: Relaxations and Convexification–Part II,” *IEEE Transactions on Power Systems*, vol. 28, no. 3, pp. 2565-2572, August 2013.