Approximate Representation of ZIP Loads in a Semidefinite Relaxation of the OPF Problem

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***Abstract*—Recent research has applied semidefinite programming (SDP) to the optimal power flow (OPF) problem. Extending SDP formulations to include the ZIP load model, which consists of constant impedance, constant current, and constant power components, is necessary for many practical problems. Existing SDP formulations consider constant power components and can be easily extended to incorporate constant impedance components. With linear dependence on voltage magnitude, constant current components are not trivially formulated in a manner amenable to SDP. This letter details an approximate representation of ZIP loads in a SDP relaxation of the OPF problem.**

***Index Terms*—ZIP load model, Semidefinite programming**

I. Introduction

POWER system analyses benefit from flexible and detailed representation of load behavior; specifying load models to best capture physical behavior is an active research topic [1], [2]. Static analyses often use ZIP load models which consist of constant impedance, constant current, and constant power components. [2], [3]. Many commercial software packages (e.g., PSS/E, PSLF, and PowerWorld) model ZIP loads.

ZIP load models are often used in the optimal power flow (OPF) problem. The non-convex OPF problem determines an optimal operating point for an electric power system in terms of a specified objective function, subject to both network equality constraints and engineering limits.

Using a rank relaxation, recent research has applied semidefinite programming (SDP) to convexify the OPF problem [4]. If the relaxed problem satisfies a rank condition, a global optimum of the OPF problem can be determined in polynomial time. No prior OPF solution method guarantees calculation of the global solution in polynomial time; SDP thus has a substantial advantage over other solution techniques. However, the rank condition is not always satisfied, so the SDP solution may not be physically meaningful [5].

Existing SDP formulations of the OPF problem explicitly consider constant power load models, in which complex power demand is independent of voltage magnitude. Constant impedance loads, for which demands are functions of the square of voltage magnitudes, are easily incorporated in existing SDP formulations using shunt admittances. However, constant current loads are linear functions of voltage magnitude, and hence are not easily incorporated into SDP formulations.

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Using a rank relaxation to approximate a linear function of voltage magnitude, this letter presents an approximate representation of constant current loads, and therefore ZIP models, for a SDP relaxation of the OPF problem.

II. The ZIP Load Model

The constant impedance, constant current, and constant power components of a ZIP load are represented by a second-order polynomial in bus voltage magnitude V_i [2], [3].

$$P_{Di}\left(V_i\right) = a_{1i}V_i^2 + a_{2i}V_i + a_{3i} \tag{1a}$$

$$Q_{Di}\left(V_i\right) = b_{1i}V_i^2 + b_{2i}V_i + b_{3i} \tag{1b}$$

where a_{1i}, a_{2i}, a_{3i} and b_{1i}, b_{2i}, b_{3i} are scalar parameters for bus i active and reactive power demand. The constant impedance ("Z"), constant current ("I"), and constant power ("P") components are specified using a_{1i} and b_{1i}, a_{2i} and b_{2i}, and a_{3i} and b_{3i}, respectively. This model forms the right hand side of the power balance equations of an OPF problem.

III. A SDP Formulation of the ZIP Load Model

The ZIP load model is composed of constant, linear, and square functions of voltage magnitude. Consider a rank one matrix $\mathbf{\Gamma}_i$ to represent these terms at bus i.

$$\mathbf{\Gamma}_i = \begin{bmatrix} 1 \\ V_i \end{bmatrix} \begin{bmatrix} 1 & V_i \end{bmatrix} = \begin{bmatrix} 1 & V_i \\ V_i & V_i^2 \end{bmatrix} \tag{2}$$

Let matrix superscripts denote the corresponding (row, column) element. With constraints $\mathbf{\Gamma}_i^{11} = 1$ and $\mathbf{\Gamma}_i^{22} = V_i^2$, linear functions of voltage magnitude are obtained using $\mathbf{\Gamma}_i^{12}$. (Squared voltage magnitudes V_i^2 are easily formulated in the SDP relaxation of the OPF problem.)

To form a SDP-suitable convex relaxation, the rank one condition on $\mathbf{\Gamma}_i$ is replaced by a positive semidefinite constraint $\mathbf{\Gamma}_i \succeq 0$. This relaxation upper bounds a convex feasible space in the $\mathbf{\Gamma}_i^{22}$ vs. $\mathbf{\Gamma}_i^{12}$ plane on the curve $\mathbf{\Gamma}_i^{12} = \sqrt{\mathbf{\Gamma}_i^{22}}$. Rather than necessarily lying on this curve, the variables $\mathbf{\Gamma}_i^{12}$ and $\mathbf{\Gamma}_i^{22}$ must be within this feasible space, which is shown in Fig. 1. An exact solution is obtained when $\text{rank}\left(\mathbf{\Gamma}_i\right) = 1$.

To enforce voltage magnitudes between V_i^{min} and V_i^{max}, constrain $\left(V_i^{min}\right)^2 \leq \mathbf{\Gamma}_i^{22} \leq \left(V_i^{max}\right)^2$. The line connecting the points $\left(\left(V_i^{min}\right)^2, V_i^{min}\right)$ and $\left(\left(V_i^{max}\right)^2, V_i^{max}\right)$ lower bounds the feasible space, which is the convex hull of the square root function between the voltage magnitude limits.

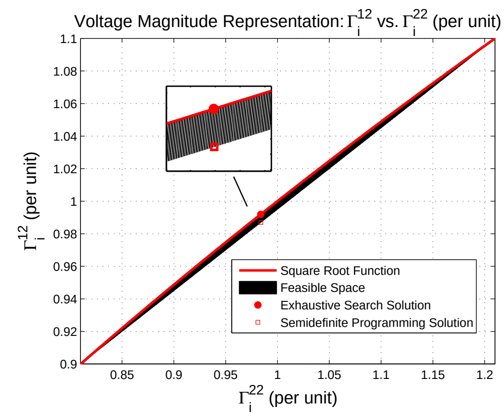

Fig. 1. Feasible Space for Voltage Magnitude Representation

To represent the ZIP load model in terms of $\mathbf{\Gamma}_i$, define

$$\mathbf{R}_i = \begin{bmatrix} a_{3i} & \frac{a_{2i}}{2} \\ \frac{a_{2i}}{2} & a_{1i} \end{bmatrix} \quad (3a)$$

$$\mathbf{T}_i = \begin{bmatrix} b_{3i} & \frac{b_{2i}}{2} \\ \frac{b_{2i}}{2} & b_{1i} \end{bmatrix} \quad (3b)$$

The proposed representation for ZIP loads is then

$$P_{Di} = \text{trace}\left(\mathbf{R}_i \mathbf{\Gamma}_i\right) \quad (4a)$$

$$Q_{Di} = \text{trace}\left(\mathbf{T}_i \mathbf{\Gamma}_i\right) \quad (4b)$$

IV. Example

Consider a two-bus system with a generator at bus 1 that has no generation limits and a ZIP load at bus 2. The line has impedance of $0.05 + j0.15$ per unit and has no flow limit. With a 100 MVA base, the ZIP load has per unit parameters $a_{12} = b_{12} = 0.01$, $a_{22} = b_{22} = 0.02$, and $a_{32} = b_{32} = 0.50$. Bus voltage magnitudes are in the range $[0.90, 1.10]$ per unit. Denote the bus i voltage phasor as $V_{di} + jV_{qi}$ per unit.

Define an OPF problem that minimizes the active power generation by specifying a \$1/MWh generation cost. Use bus 1 to provide the reference angle (i.e., $V_{q1} = 0$ and $V_{d1} = V_1$). For specified V_1, the feasible space has two degrees of freedom (i.e., V_{d2} and V_{q2}) constrained by the active and reactive power balance equations at bus 2. Exhaustive search of the feasible space is conducted by varying V_1 between 0.90 and 1.10 per unit while solving for V_{d2} and V_{q2} with the quadratic equation. This yields a globally optimal solution to the OPF problem with objective value of \$55.82, $V_1 = 1.100$, and $V_2 = 0.992$ per unit ($V_{d2} = 0.991$ and $V_{q2} = -0.048$ per unit), which is marked by the circle in Fig. 1.

The solution to the SDP relaxation closely approximates this global solution obtained through exhaustive search. The solution to the SDP relaxation has optimal objective value \$55.81 and $V_1 = 1.100$ per unit. $\mathbf{\Gamma}_2^{22} = 0.984$, which implies that $V_2 = 0.992$, while $\mathbf{\Gamma}_2^{12}$ implies that $V_2 = 0.987$ (i.e., a voltage magnitude difference of 0.005 per unit or 0.5%). This solution, which is marked by the square in Fig. 1, indicates that the SDP relaxation selects a slightly larger value for $\mathbf{\Gamma}_2^{22}$ in order to minimize losses as compared to a smaller value for $\mathbf{\Gamma}_2^{12}$ to reduce the demand of the ZIP load. The matrix $\mathbf{\Gamma}_2$ has eigenvalues of 0.005 and 1.979, so this matrix is close to being rank one. The proposed model thus closely approximates but does not exactly match ZIP load behavior for this system.

V. Discussion

Reducing active power demand at ZIP loads often results in lower cost solutions to OPF problems. Active power demands of the constant current components of ZIP loads with positive a_{2i} are reduced by minimizing voltage magnitudes. Thus, solutions to the SDP relaxation will tend to have smaller values of $\mathbf{\Gamma}_i^{12}$ relative to the value of V_i implied by $\sqrt{\mathbf{\Gamma}_i^{22}}$, which results in rank two $\mathbf{\Gamma}_i$ matrices. Although such solutions are not exact, they closely approximate ZIP load behavior. The maximum error in the voltage magnitude approximation is

$$\max\left\{\sqrt{\mathbf{\Gamma}_i^{22}} - \mathbf{\Gamma}_i^{12}\right\} = \frac{\left(V_i^{max} - V_i^{min}\right)^2}{4\left(V_i^{max} + V_i^{min}\right)} \quad \text{per unit} \quad (5)$$

This maximum occurs at $\mathbf{\Gamma}_i^{22} = \left(V_i^{max} + V_i^{min}\right)^2/4$. The error is small for typical values of V_i^{max} and V_i^{min} (e.g., a maximum error of 0.005 per unit occurs at $\mathbf{\Gamma}_i^{22} = 1.00$ and $\mathbf{\Gamma}_i^{12} = 0.995$ per unit for $V_i^{max} = 1.10$ and $V_i^{min} = 0.90$).

Note that negative a_{2i} values in ZIP load models correspond to constant current components that inject active power into the network. For these cases, the SDP relaxation will typically yield an exact solution because it tends to maximize the magnitudes of these negative injections. For instance, specifying $a_{22} = -0.02$ per unit gives the exact solution to the two-bus example system (i.e., $\text{rank}\left(\mathbf{\Gamma}_2\right) = 1$).

Also note that some non-zero values for a_{2i} and b_{2i} may result in failure to satisfy the rank condition of [4]. Experience suggests that this is more common with larger positive a_{2i}.

Numerical experience included cases ranging in size up to the IEEE 118-bus system. With ZIP loads at each bus, solver times increased by approximately 20% to 30% relative to cases with only constant power loads. The percent increases in solution times are relatively independent of system size.

VI. Conclusion

This letter presents an approximate method for incorporating ZIP load models in a SDP formulation of the OPF problem.

References

[1] D. Kosterev, A. Meklin, J. Undrill, B. Lesieutre, W. Price, D. Chassin, R. Bravo, and S. Yang, "Load Modeling in Power System Studies: WECC Progress Update," in *Power and Energy Society General Meeting - Conversion and Delivery of Electrical Energy in the 21st Century, 2008 IEEE*, July 2008, pp. 1–8.

[2] F. Chassin, E. Mayhorn, M. Elizondo, and S. Lu, "Load Modeling and Calibration Techniques for Power System Studies," in *North American Power Symposium (NAPS), 2011*, August 2011, pp. 1–7.

[3] P. Kundur, N. Balu, and M. Lauby, *Power System Stability and Control*. McGraw-Hill Professional, 1994.

[4] J. Lavaei and S. Low, "Zero Duality Gap in Optimal Power Flow Problem," *IEEE Transactions on Power Systems*, vol. 27, no. 1, pp. 92–107, February 2012.

[5] B. C. Lesieutre, D. K. Molzahn, A. R. Borden, and C. L. DeMarco, "Examining the Limits of the Application of Semidefinite Programming to Power Flow Problems," in *49th Annual Allerton Conference on Communication, Control, and Computing, 2011*, September 28-30 2011.