Proof of the Inexactness of Second-order Cone Relaxations for Calculating Operating Envelopes

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***Abstract*—Rapidly growing deployments of distributed energy resources like rooftop solar and electric vehicles have disrupted the status quo of distribution network operations. Power export/import limits that change as network conditions change, also known as operating envelopes, have been proposed as a means of safely operating distribution networks containing distributed energy resources. This paper analyzes the impacts of using convex relaxations, specifically the second-order cone relaxation, for determining the limits required for safe operation. Building on prior work, we illustrate the inadequacy of operating envelopes calculated using a second-order cone formulation for ensuring network safety. Then we discuss the complexity in the relationship between the relaxation, the nonlinear formulation, and the network parameters. For the case of purely resistive, radial networks and the case of purely reactive, radial networks, we prove that the second-order cone relaxation is inexact and exact, respectively, for the operating envelope problem. This work illustrates that while using a second-order cone relaxation to determine operating envelopes will not always result in overly large bounds, it often will and with potentially damaging consequences.**

***Index Terms*—Distributed Energy Resources, Operating Envelopes, Second-order Cone Power Flow Relaxation.**

I. Introduction

Proliferation of small-scale distributed energy resources (DERs), such as roof-top solar photovoltaics (PV), batteries, and controllable loads, within distribution networks has become a key focus of power systems research [1]–[3]. Power injections or large load changes caused by DERs can lead to operational violations like over- and under-voltages, harmonics, frequency distortions, faults, and protection problems [4]–[6]. A wide variety of solutions have been proposed to maintain safe operations in distribution networks containing active DERs and DER aggregations, including coordination strategies between aggregators and distribution network operators [7], constructing a constraint set on aggregator controls [8], and constructing a set of nodal power injection limits, also known as operating envelopes or dynamic hosting capacity [9]–[13].

Dynamic operating envelopes, or simply operating envelopes, represent time-varying limits on power imports and/or exports at each active bus in the distribution network [13]. An active bus is defined as any bus with a connected DER capable of generating or consuming power. To calculate operating envelopes, it is common to solve a modified version of the optimal power flow (OPF) problem. Originally formulated to find the lowest-cost generator dispatch which satisfies network constraints [14], [15], the OPF problem can refer to any optimization problem subject to the power flow equations and other operational constraints [16]. The nonlinear AC power flow equations make the OPF problem non-convex, i.e., a solution is not guaranteed to be globally optimal and can be computationally challenging for large networks. The use of convex relaxations or approximations of the power flow equations can help alleviate these issues. However, relaxations do not always give exact or feasible solutions [17].

There is extensive literature on relaxations, including second-order cone (SOC) relaxations, for the OPF problem [18]–[24]. Existing work focuses on OPF problems with objectives such as minimizing generation costs or minimizing losses. In contrast, the operating envelope OPF problem aims to maximize the region of feasible power injections. To the best of our knowledge, there has been no prior work exploring the exactness of a SOC relaxation for a problem of this type, aside from our prior conference paper [25], which this paper significantly extends, and the illustrations of SOC solutions that cannot be physically realized in [9], [26]. More discussion of existing literature on SOC relaxations and its relevance for the operating envelope problem is provided in Section III.

The contribution of this paper is an exploration of conditions for which the SOC relaxation is and is not exact for the operating envelope problem. As described above, existing studies have already shown, empirically, that the SOC relaxation is (usually) not exact for the operating envelope problem. In our prior work [25], we posited that the flexibility introduced by the SOC relaxation allows for artificial increases in the squared current. Larger squared current values create larger power losses which in turn permit artificially large operating envelope upper limits. Based on this idea, we proposed a modification to the objective function of the SOC formulation (specifically, we penalize power losses) to achieve exactness [25]. Here, we use a case study to illustrate that while this modification may lead to an exact SOC relaxation (for the penalized problem) or at least an AC power flow-feasible solution, it is not guaranteed to return the optimal solution to the original problem. Then, we analyze the unmodified SOC formulation and we show that, surprisingly, there exist parameter values for which the SOC relaxation is exact for the operating envelope problem. We illustrate analytically that identifying conditions under which the relaxation is guaranteed to be exact or guaranteed to be inexact is not simple. Finally, we prove that the SOC relaxation of the operating envelope problem will not be exact for purely resistive, radial networks if at least one bus in the network is

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below its upper voltage limit at optimality of the nonlinear formulation. We also prove that the SOC relaxation is exact for the operating envelope problem for purely reactive, radial networks. For realistic systems, with both positive resistance and positive reactance, we are unable to establish simple conditions on SOC relaxation exactness. For this reason, we recommend not using the SOC relaxation when computing operating envelopes.

Section II presents the paper's notation and OPF formulations for calculating operating envelopes. The modified objective function and a case study illustrating the impacts of the SOC relaxation are discussed in Section III. Section IV investigates conditions for which the SOC relaxation is and is not exact for the operating envelope problem. Conclusions are given in Section V.

II. Problem Formulations

Many different approaches and optimization formulations have been used to compute operating envelopes. One option for including the power flow equations is to use the nonlinear AC power flow; however, this requires the assumption that any power injection between the operating envelope lower (import) and upper (export) limit will not lead to operational constraint violations. Another interpretation of this assumption is that the relationship between voltage and power injections is monotonic. The authors of [27] show that this assumption does not always hold, particularly in unbalanced networks. A second option is to use a power flow linearization to improve computation time and avoid convergence issues [27], e.g., as in [10], [11], [28], [29]. However, linearizations come with a loss of model accuracy and (most) come with no guarantees on performance/accuracy or solution feasibility. A third option is to use a convex relaxation, such as an SOC [12] or quadratically constrained [30] formulation, which can also improve computation time and convergence versus solving a nonlinear AC power flow problem. Convex relaxations have advantages over linearizations in that relaxation solutions that are feasible for the AC power flow constraints are necessarily optimal in the AC power flow problem. However, since the feasible region may extend beyond the AC power flow feasible region, relaxations may admit solutions that violate network constraints. Therefore, the remainder of this paper will argue why relaxations should *not* be used to compute operating envelopes. A fourth option is to use a convex restriction or inner approximation of the feasible region [9], [27]. In contrast to convex relaxations, the feasible region is contained within the AC power flow feasible region. However, operating envelopes are usually calculated by solving two problems, one for the lower limit and one for the upper limit. While the feasible region of each problem is convex, the union of the two feasible regions is not necessarily convex. One could either make a monotonicity assumption or find a convex restriction of the union of the feasible regions.

In addition to the choice of power flow formulation, one also needs to choose an appropriate objection function. One choice of objective function is to maximize aggregate flexibility, e.g., the sum of the operating envelopes across the network. In addition to maximizing flexibility, it is also important to consider fairness when computing operating envelopes [11]–[13], [31]. When the objective is to maximize the sum of operating envelopes across the feeder, buses further down the feeder tend to receive significantly tighter limits [13]. This means that a customer with a DER who is far from the substation will have less ability to inject power and/or modify load than a customer who is close to the substation. Therefore, this paper utilizes the objective function proposed in [12], which maximizes the smallest operating envelope. A version of the formulation presented in this paper with the objective of maximizing the sum of the operating envelopes across the feeder was also analyzed and produced similar findings. Those results are omitted for brevity.

A. Operating Envelope AC-OPF Branch Flow Formulation

We first define the full nonlinear operating envelope problem, which incorporates the AC power flow equations. This formulation is based on [12] and used in [25]. Different from our formulation in [25], we consider a more flexible case for reactive power injections in this work, as detailed below.

Consider a radial distribution network with a set of buses $\mathcal{N}$ and a set of lines $\mathcal{L}$. Let bus 0 denote the substation. In a radial network, each bus i has exactly one parent bus, i.e., the bus immediately upstream, which we denote as i', and a set of buses immediately downstream, which we denote as $\mathcal{C}^i$. For simplicity, we assume the network is balanced and can be represented by its single-phase equivalent circuit. We use the branch flow model for representing power flow [18], where the voltage and current angles can be omitted by writing the power flow and voltage difference equations in terms of squared voltage magnitudes and squared current magnitudes. Let $z_{i'i} = r_{i'i} + jx_{i'i}$ represent the impedance of the line connecting buses i' and i. The apparent power flow limit on the line connecting buses i' and i is $\overline{s}_{i'i}$. The per unit (p.u.) squared voltage limits at each bus are $\underline{v}$ and $\overline{v}$.

We describe in detail the problem to compute export limits; import limits are computed similarly. The decision variables are p_i^{exp}, $\forall i \in \mathcal{N}$, the net active power export limits (i.e., the upper limit of the operating envelope) at all buses $i \in \mathcal{N}$; $\underline{p}^{\text{exp}}$, the smallest export limit in the network; $p_{i'i}\,\forall i'i \in \mathcal{L}$, the active power flowing from bus i' to bus i on all branches $i'i \in \mathcal{L}$; $q_{i'i}\,\forall i'i \in \mathcal{L}$, the reactive power flowing from bus i' to bus i on all branches $i'i \in \mathcal{L}$; $v_i\,\forall i \in \mathcal{N}$, the squared voltage magnitude at all buses $i \in \mathcal{N}$; and $l_{i'i}\,\forall i'i \in \mathcal{L}$, the squared current magnitude on all lines $i'i \in \mathcal{L}$. We assume that the reactive power injections at each node, q_i^{exp}, are controllable such that the power factor at each bus is at least 0.95. Let $\mathbf{x}$ be a vector of stacked decision variables. We compute the operating envelopes by solving:

$$\max_{\mathbf{x}} \ \underline{p}^{\text{exp}} \tag{1a}$$

$$\text{s.t.} \ \underline{p}^{\text{exp}} \le p_i^{\text{exp}}, \quad \forall i \in \mathcal{N} \tag{1b}$$

$$\sum_{j \in \mathcal{C}^i} p_{ij} = p_{i'i} - r_{i'i} l_{i'i} + p_i^{\text{exp}}, \ \forall i \in \mathcal{N} \tag{1c}$$

$$\sum_{j \in \mathcal{C}^i} q_{ij} = q_{i'i} - x_{i'i} l_{i'i} + q_i^{\text{exp}}, \ \forall i \in \mathcal{N} \tag{1d}$$

$$-p_i^{\exp}\alpha^{\rm pf} \leq q_i^{\exp} \leq p_i^{\exp}\alpha^{\rm pf}, \quad \forall i \in \mathcal{N} \tag{1e}$$

$$v_i = v_{i'} - 2(r_{i'i}p_{i'i} + x_{i'i}q_{i'i}) + (r_{i'i}^2 + x_{i'i}^2)l_{i'i}, \quad \forall i \in \mathcal{N} \tag{1f}$$

$$p_{i'i}^2 + q_{i'i}^2 = l_{i'i}v_{i'}, \quad \forall i'i \in \mathcal{L} \tag{1g}$$

$$p_{i'i}^2 + q_{i'i}^2 \leq \overline{s}_{i'i}^2, \quad \forall i'i \in \mathcal{L} \tag{1h}$$

$$\underline{v} \leq v_i \leq \overline{v}, \quad \forall i \in \mathcal{N} \tag{1i}$$

$$v_0 = 1, \tag{1j}$$

$$l_{i'i} \geq 0, \quad \forall i'i \in \mathcal{L} \tag{1k}$$

where $\alpha^{\rm pf} = \tan(\cos^{-1}(0.95))$.

Constraint (1b) converts a $\max\min$ objective into a linear one by defining the smallest export limit to be maximized. Constraints (1c) and (1d) enforce active and reactive power balance at each bus. Constraint (1e) ensures that the power factor at each bus is at least 0.95. Constraint (1f) defines the squared voltage at each bus. Constraint (1g) defines the squared apparent power flowing from bus i' to i and (1h) limits it. Constraint (1i) enforces the voltage limits at each bus and (1j) fixes the substation voltage to 1 p.u. Lastly, (1k) requires the squared current to be nonnegative. This formulation assumes monotonicity, i.e., if $p^{\exp}$ defines the upper bound of a valid operating envelope, then any injection less than $p^{\exp}$, down to a similarly defined lower bound (import limit), is feasible. We note that this may not always be true in practice since (1g) is non-convex.

B. Second-Order Cone Relaxation

The SOC relaxation was first proposed for the branch flow model in [18] and proven in [32] to be equivalent to the SOC relaxation of the bus injection model (first proposed in [33]). The SOC formulation is obtained from the full nonlinear formulation (1) by relaxing the equality in (1g) to $p_{i'i}^2 + q_{i'i}^2 \leq l_{i'i}v_{i'}$. This can be equivalently written in a standard SOC form as

$$\left\| \begin{matrix} 2p_{i'i} \\ 2q_{i'i} \\ l_{i'i} - v_{i'} \end{matrix} \right\|_2 \leq l_{i'i} + v_{i'}, \qquad \forall i'i \in \mathcal{L}. \tag{2}$$

The SOC formulation of the operating envelope problem is

$$\max \underline{p}^{\exp} \quad \text{s.t. (1b)} - \text{(1f)},\ \text{(1h)} - \text{(1k)},\ \text{(2)}. \tag{3}$$

The solution given by this formulation is *exact*, i.e., it is the solution to the nonlinear problem, if it is feasible in the nonlinear problem, i.e., if $p_{i'i}^2 + q_{i'i}^2 = l_{i'i}v_{i'}$ holds for every $i'i \in \mathcal{L}$. In the remainder of this work we use the difference between the left and right sides of this equation to quantify the solution's inexactness.

While successfully used for a variety of OPF problems, the SOC relaxation is problematic when used in operating envelope problems, as we will show next, and as has been shown previously in [25], [26].

III. SOC Inexactness

In this section, we demonstrate the issues associated with using a SOC relaxation to construct operating envelopes. First, we describe an objective function modification proposed in our prior work [25]. Then, we present a case study that demonstrates SOC inexactness within the operating envelope problem, reinforcing previous observations. The case study also reveals an issue with the objective function modification.

A. Objective Function Modification

In previous work [25], we presented a modification of the objective function for improving the efficacy of operating envelopes calculated using SOC relaxations of the power flow equations. In [25], case studies illustrated how the SOC formulation can permit larger operating envelopes, which fail to ensure acceptable voltages under the true (nonlinear) physics. In an attempt to alleviate issues believed to be caused by fictitious flexibility in current values introduced by the relaxation, we penalized power losses in the objective [25]. Specifically, the modified objective that we presented was

$$\max_{\mathbf{x}} \ \underline{p}^{\exp} - \lambda \sum_{i'i \in \mathcal{L}} l_{i'i}r_{i'i}, \tag{4}$$

where λ is a weighting parameter found heuristically to account for differences in magnitude between the two terms. A discussion regarding the tuning of λ is presented in [25].

In [25], we presented two case studies for which this objective function modification led to solutions that 1) closely matched the solution found using the nonlinear formulation, and 2) had negligible values for inexactness. The networks used in those two case studies were relatively small, containing only 4 and 56 buses, respectively.

B. Case Study Setup

In this section, we use a larger and more realistic network than those considered in [25] to highlight the implications of using the SOC relaxation in the operating envelope problem and also the implications of using the modified objective function. Specifically, we use the 141-bus network MATPOWER test case [34]. Note that the substation is considered to be bus 0 and bus 1 is the bus connected to the substation. For simplicity, we assume that every bus needs an operating envelope. Voltages between 0.95 and 1.05 p.u. are considered acceptable. Any operation that leads to values below or above these limits will be considered unacceptable.

To show how the SOC relaxation of the power flow equations and the objective function modification impact the calculation of operating envelopes, we compare the results from both formulations to the full nonlinear operating envelope formulation. The solutions are compared in terms of the size of the obtained operating envelopes and the efficacy of those operating envelopes, i.e., how well they maintain acceptable voltages. To analyze the efficacy of the resulting operating envelopes, the nonlinear power flow equations are solved for each formulation assuming net power injections at each bus are equal to the upper limit set by the operating envelopes. Note that because the relationship between voltage and power injections is not monotonic, or even convex, this is not guaranteed to result in the worst case voltages for any injections within the operating envelopes. However, as described in Section II, it is commonly assumed that any injections within the operating envelope are permissible and will lead to acceptable voltages.

Both the nonlinear and SOC problems were solved in Julia using the JuMP package [35]. The solvers used were IPOPT 3.14.4 [36] and CPLEX 20.1.0 for the nonlinear and SOC problems, respectively.

C. Case Study Results

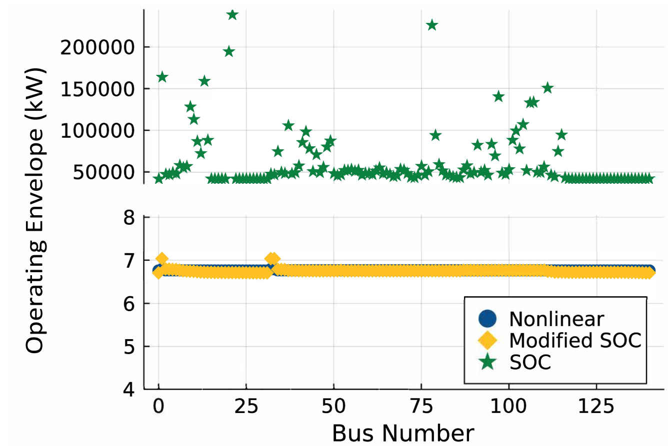
We find that, using the full nonlinear operating envelope formulation, the operating envelope upper limits varied between 6.76 and 6.79 kW, shown as dark blue circles in Fig. 1a. When every bus is injecting real power equal to these limits, the voltages at each bus correspond to the values shown in dark blue in Fig. 1b. As can be seen in the figure, the voltage at every bus is within the voltage limits. A few of the buses far away from the substation are at the upper voltage limit, but this is to be expected at the edge of safe operation.

When the SOC relaxation is used with the same objective function, the solution found gives operating envelope upper limits that vary between 42.05 and 238.74 MW, shown as green stars in Fig. 1a. This range contains values 4 or 5 orders of magnitude larger than the range given by the nonlinear formulation. These power injection limits are so large, the AC power flow solver failed to converge when bus injections were set equal to them. The limits calculated from the SOC relaxation are so comparatively high that they are essentially equivalent to implementing no injection limits, and would likely lead to severe over-voltages, damaged equipment, and possible system instability if implemented.

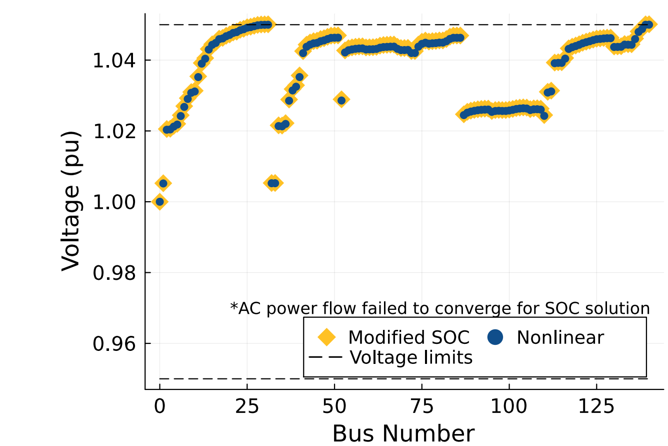
When the SOC relaxation is used with the modified objective function (with $\lambda = 0.04$, where this choice will be explained below), the resulting operating envelopes have upper limits between 6.71 and 7.03 kW. While these limits are significantly closer to the limits given by the nonlinear formulation than those given by the unmodified SOC formulation, they are far from a perfect match. However, Fig. 1b shows that the bus voltages resulting from solving AC power flow when net power injections are equal to the operating envelopes generated via the modified SOC formulation are similar to the voltages given by the nonlinear case, and more importantly, the voltages are all safe. This means that the SOC formulation with the objective function modification can lead to safe operating envelopes. However, in more complex networks, the solution obtained will likely be different from the nonlinear solution.

Table I provides a numerical comparison between the solutions from the nonlinear formulation, the SOC formulation, and the SOC formulation using the modified objective function with $\lambda = 0.04$ (Mod SOC). Values $\underline{p}^{\text{exp}}$ represent the smallest operating envelope upper limit in the network for each case. Max Bus Inexactness values represent $\max_{i'i \in \mathcal{L}} |p_{i'i}^2 + q_{i'i}^2 - l_{i'i} v_{i'}|$ and the Total Inexactness values represent $\sum_{i'i \in \mathcal{L}} |p_{i'i}^2 + q_{i'i}^2 - l_{i'i} v_{i'}|$. The results highlight the improvements that the objective function modification have on the inexactness of the SOC formulation. However, pairing these results with the results in Fig. 1b tells us that even if a SOC solution does not satisfy (1g), i.e., the SOC is not exact, it can still result in safe voltages. This suggests that the SOC relaxation with modified objective function could be used to determine operating envelopes; however, the corresponding voltage, current, and power solution values are unreliable and likely not physically realizable.

A fundamental challenge to using the objection function modification is how to select λ, as we discussed in [25]. Figures 2 and 3 illustrate that selecting the "best" λ may not be easy, even with knowledge of the solution to the nonlinear formulation. Without the nonlinear solution, the measure of inexactness could be used to select λ. However, unlike in the small network cases in [25] where the sum of inexactness



(a) The operating envelopes found using the nonlinear, SOC, and modified SOC formulations with $\lambda = 0.04$.



(b) The voltage at each bus resulting from solving AC power flow with power exports at each bus set equal to the operating envelope limits calculated using the nonlinear and modified SOC formulations with $\lambda = 0.04$. Note that voltages for the SOC formulation are not shown because the AC power flow solver failed to converge.

Fig. 1. Results for the 141-bus network using nonlinear, SOC, and modified SOC formulations.

TABLE I
MEASURES OF INEXACTNESS

	Nonlinear	SOC	Mod SOC
$\underline{p}^{\text{exp}}$ (kW)	6.76	42052	6.71
Max Bus Inexactness (MVA^2)	–	44032	33.83
Total Inexactness (MVA^2)	–	152667	33.95

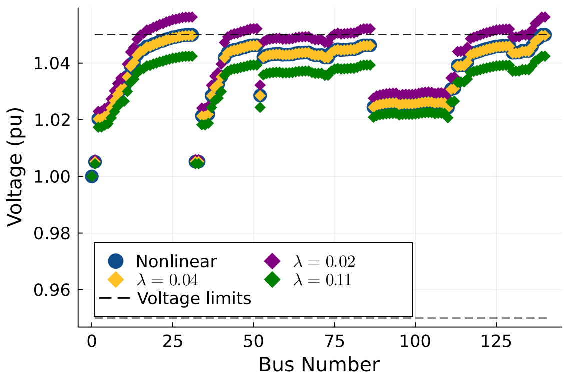


Fig. 2. The impact of λ. Like in Fig. 1b, we plot the voltage at each bus in the 141-bus network resulting from solving AC power flow with power exports at each bus equal to the operating envelope limits calculated using the nonlinear and modified SOC formulations, but for three choices of λ.

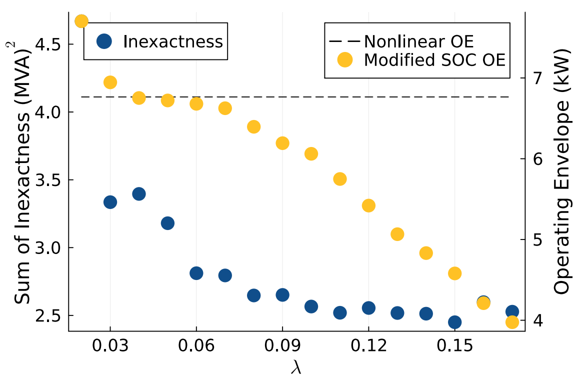


Fig. 3. The sum of the inexactness, i.e., $lv - (p^2 + q^2)$, at every bus, and the smallest resulting operating envelope as λ is increased.

converged to zero as the operating envelope size approached the nonlinear solution, here the sum of inexactness remains significant even when the smallest operating envelope becomes smaller than the smallest operating envelope given by the nonlinear solution. In Figs. 1a and 1b and Table I, we used $\lambda = 0.04$ despite the relatively high inexactness shown in Fig. 3 because it led to the largest operating envelopes that maintained safe voltage levels. In practice, one could select an appropriate λ by checking the voltage levels given by AC power flow with power exports at each bus set equal to the operating envelope limits.

IV. Conditions for Inexactness

We have shown that the solutions to operating envelope problems using SOC relaxations are not always effective at maintaining acceptable voltages. Prior literature on the topic of SOC relaxations for the operating envelope problem suggests that the relaxation will never be exact [9], [25], [26]. If this can be proved mathematically, it would justify the argument that SOC relaxations should never be used for the operating envelope problem, or similar problems. Alternatively, if conditions can be found under which the relaxation is exact, then use of the SOC relaxation may be justified for those cases.

There has been significant work in the area of SOC relaxations for distribution networks, including work on distribution networks containing DERs. It is proven in [18], [37] that a SOC relaxation is exact when minimizing an objective function that is strictly increasing in line losses if over-satisfaction of load is allowed. In [38], it is shown that a SOC relaxation is exact for an OPF problem in a radial distribution network if there are no upper bounds on voltage, voltage magnitudes are roughly 1 p.u., $\frac{r}{x}$ is bounded, and net loads are below 1 p.u. This is shown under the condition that the objective function is strictly increasing with respect to the active power injections at the substation. In [39], it is shown that for a radial network, the SOC relaxation is exact for an objective function minimizing an increasing convex function over the power injection region. It is shown in [22] that the SOC relaxation of an OPF problem minimizing a convex objective function will be exact if no upper voltage limits are binding at optimality.

None of the above proofs apply to the operating envelope problem because, as the authors of [40] note, the sets of sufficient conditions for an exact SOC relaxation always include “a requirement on the objective to be a minimization of a function increasing with the branch flow apparent powers.” Furthermore, [40] explains how these types of objective functions are at odds with what is expected of an active distribution system and provides a case study illustrating that including DER-centric objectives in the objective function can lead to an inexact SOC relaxation. They then present a cutting-plane algorithm to tighten the relaxation by iteratively decreasing losses. This approach has since been applied to the hosting capacity problem [41].

In this section, we show that there actually are cases for which the SOC relaxation is exact for the operating envelope problem. We then use numerical analysis to determine the conditions under which the SOC relaxation is or is not exact. Finally, we prove mathematically that for radial networks the SOC relaxation is not exact for the operating envelope problem given a purely resistive network, and it is exact given a purely reactive network.

A. Conditions for an exact relaxation

In this subsection, we numerically analyze conditions that lead to an exact or inexact SOC relaxation of the operating envelope problem. Specifically, we explore the impacts of line parameters r and x on the optimal solution of both the nonlinear and SOC formulations. We do this by rewriting the SOC formulation in terms of the optimal solution of the nonlinear formulation and defining new variables that represent the deviations from the nonlinear variables at optimality. We then define a feasibility problem, which if feasible indicates that the SOC relaxation is not exact. We solve this feasibility problem and the rewritten SOC formulation for different r and x values and analyze the results.

For simplicity, we consider the operating envelope problem that does not consider fairness, i.e., the nonlinear form is

$$\max_{\mathbf{x}} \sum_{i \in \mathcal{N}} p_i^{\text{exp}} \quad \text{s.t.} \quad (1c) - (1k), \tag{5}$$

and the SOC form is

$$\max_{\mathbf{x}} \sum_{i \in \mathcal{N}} p_i^{\text{exp}} \quad \text{s.t.} \quad (1c) - (1f),\ (1h) - (1k),\ (2). \tag{6}$$

Let $\mathbf{x}^* = (\mathbf{p}^{\exp *}, \mathbf{q}^{\exp *}, \mathbf{p}^*, \mathbf{q}^*, \mathbf{l}^*, \mathbf{v}^*)$ be the optimal solution to (5). Consider a candidate solution for (6) of the form $\hat{\mathbf{x}} = \mathbf{x}^* + \delta$ where δ represents any deviation away from the optimal solution $\mathbf{x}^*$ of (5). The SOC formulation can be rewritten as

$$\max_{\delta} \quad \sum_{i \in \mathcal{N}} \delta_i^{\mathrm{p,exp}} \tag{7a}$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{C}^i} \delta_{ij}^{\mathrm{p}} = \delta_{i'i}^{\mathrm{p}} - r_{i'i}\delta_{i'i}^{\mathrm{l}} + \delta_i^{\mathrm{p,exp}}, \forall i \in \mathcal{N} \tag{7b}$$

$$\sum_{j \in \mathcal{C}^i} \delta_{ij}^{\mathrm{q}} = \delta_{i'i}^{\mathrm{q}} - x_{i'i}\delta_{i'i}^{\mathrm{l}} + \delta_i^{\mathrm{q,exp}}, \forall i \in \mathcal{N} \tag{7c}$$

$$\begin{aligned} &-(p_i^{\exp} + \delta_i^{\mathrm{p,exp}})\alpha^{\mathrm{pf}} \leq q_i^{\exp} + \delta_i^{\mathrm{q,exp}} \\ &\quad \leq \alpha^{\mathrm{pf}}(p_i^{\exp} + \delta_i^{\mathrm{p,exp}}), \qquad \forall i \in \mathcal{N} \end{aligned} \tag{7d}$$

$$\begin{aligned} &\delta_i^{\mathrm{v}} = \delta_{i'}^{\mathrm{v}} - 2(r_{i'i}\delta_{i'i}^{\mathrm{p}} + x_{i'i}\delta_{i'i}^{\mathrm{q}}) + (r_{i'i}^2 + x_{i'i}^2)\delta_{i'i}^{\mathrm{l}}, \\ &\quad \forall i \in \mathcal{N} \end{aligned} \tag{7e}$$

$$\begin{aligned} &\left\| \begin{matrix} 2p_{i'i}^* + 2\delta_{i'i}^{\mathrm{p}} \\ 2q_{i'i}^* + 2\delta_{i'i}^{\mathrm{q}} \\ l_{i'i}^* + \delta_{i'i}^{\mathrm{l}} - v_{i'}^* - \delta_{i'}^{\mathrm{v}} \end{matrix} \right\|_2 \leq l_{i'i}^* + \delta_{i'i}^{\mathrm{l}} + v_{i'}^* + \delta_{i'}^{\mathrm{v}}, \\ &\quad \forall i'i \in \mathcal{L} \end{aligned} \tag{7f}$$

$$(p_{i'i}^* + \delta_{i'i}^{\mathrm{p}})^2 + (q_{i'i}^* + \delta_{i'i}^{\mathrm{q}})^2 \leq \overline{s}_{i'i}^2, \quad \forall i'i \in \mathcal{L} \tag{7g}$$

$$\underline{v} \leq v_i^* + \delta_i^{\mathrm{v}} \leq \overline{v}, \quad \forall i \in \mathcal{N} \tag{7h}$$

$$0 = \delta_0^{\mathrm{v}} \tag{7i}$$

$$l_{i'i}^* + \delta_{i'i}^{\mathrm{l}} \geq 0, \quad \forall i'i \in \mathcal{L} \tag{7j}$$

where $\delta_i^{\mathrm{p,exp}} = \hat{p}_i^{\exp} - p_i^{\exp *}$, $\delta_{i'i}^{\mathrm{p}} = \hat{p}_{i'i} - p_{i'i}^*$, etc.

If the SOC relaxation is exact, then the problem

$$\max_{\delta} \quad \sum_{i \in \mathcal{N}} \delta_i^{\mathrm{p,exp}} \tag{8a}$$

$$\begin{aligned} \text{s.t.} \quad & (7\text{b}) - (7\text{j}), \\ & \sum_{i \in \mathcal{N} \setminus 0} \delta_i^{\mathrm{p,exp}} + \epsilon \geq 0 \end{aligned} \tag{8b}$$

should not have a solution. In (8b), $\epsilon > 0$ represents a small positive value such that the constraint actually enforces $\sum_{i \in \mathcal{N} \setminus 0} \delta_i^{\exp} > 0$.

Since our formulation considers the net power injections and does not include explicit demand parameters, the only parameters in the formulation are r, x, and the power factor considered in (7d). This motivated the idea that the conditions for which (3) is exact could be conditions on r and x. To investigate this, we solved (8) over a range of r and x values for a simple 2-bus example, the 56-bus example from [25], and the 141-bus example from Section III-B. For the 56- and 141-bus networks, we assume all lines in the network have the same r and x values. The ranges of tested r and x values are selected to span values with similar magnitude to the r and x values given in the original network models.

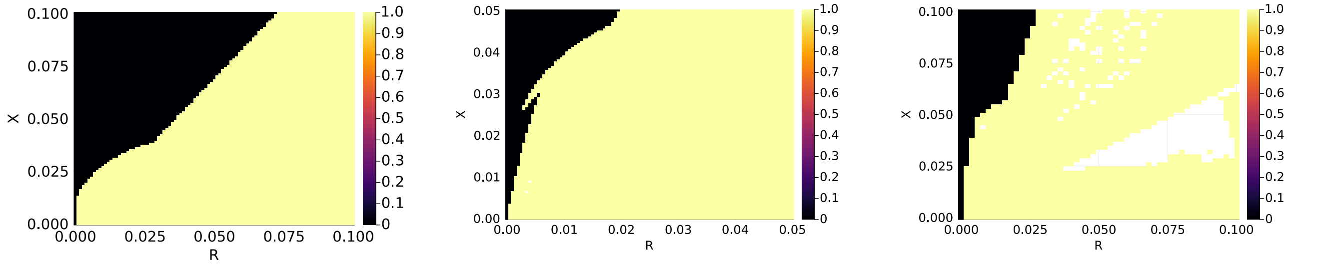
Figure 4 illustrates the feasibility of (8) for these networks over a range of r and x values. To avoid misinterpreting numerical error, we consider (8) infeasible and the SOC relaxation to be exact if it does not lead to at least a 0.1% improvement in the objective function, i.e., $\epsilon = 0.001 \left(\sum_{i \in \mathcal{N}} p_i^{\exp *}\right)$. The black regions in Fig. 4 correspond to values of r and x for which (8) is infeasible and therefore the SOC relaxation is exact. The yellow regions correspond to values of r and x for which (8) is feasible and the SOC relaxation is not exact. The white areas in Fig. 4c represent values of r and x for which (1) was (at least locally) infeasible.

The regions in the figure suggest that whether or not the SOC relaxation is exact is dependent on r and x. However, the exact relationship is unclear. Note that there is a region of r and x values for which the SOC relaxation is exact for the operating envelope problem. This is counter to previous examples where the relaxation has always been reported to be inexact [9], [25], [26]. It is also counter to the initial intuition that because operating envelopes can increase as losses increase (assuming $r > 0$), the flexibility in current values enabled by the SOC relaxation would always lead to an inexact relaxation. Figure 4 does show that when $r = 0$, i.e., there are no active power losses, the relaxation is exact for each of the three tested networks. In contrast, when the network is purely resistive, i.e., $x = 0$ and $r > 0$, the relaxation is never exact. We present a mathematical proof of both of these cases in the subsequent subsections.

Figures 5a–5f show values of δ_{01}^{p}, δ_{01}^{q}, δ_{01}^{l}, δ_1^{v}, $\delta_1^{\mathrm{p,exp}}$, and $\delta_1^{\mathrm{q,exp}}$, i.e., the change in variable values from the optimal solution to the nonlinear formulation and the optimal solution to the SOC formulation for the 2-bus network. All six plots show three regions: i) the white region where (8) is infeasible, ii) the "dome-shaped" region near the bottom left corner, and iii) the region below the diagonal in the lower right corner. Except in Fig. 5d, the dome region and the lower diagonal region have distinct patterns with respect to r and x. This suggests that the flexibility introduced by the relaxation, which allows the SOC formulation to attain higher operating envelope values, changes not only in magnitude but also in which variables are manipulated to create this flexibility as r and x change. The squared line current value, shown in Fig. 5c, has the most significant change between the nonlinear and SOC solutions, suggesting it plays a significant role in the difference in objective values. The existence of these two distinct regions and the variations within them for the simple 2-bus network suggest that there may be multiple sets of conditions under which the SOC relaxation of the operating envelope problem is not exact, and identifying precise conditions is likely difficult.

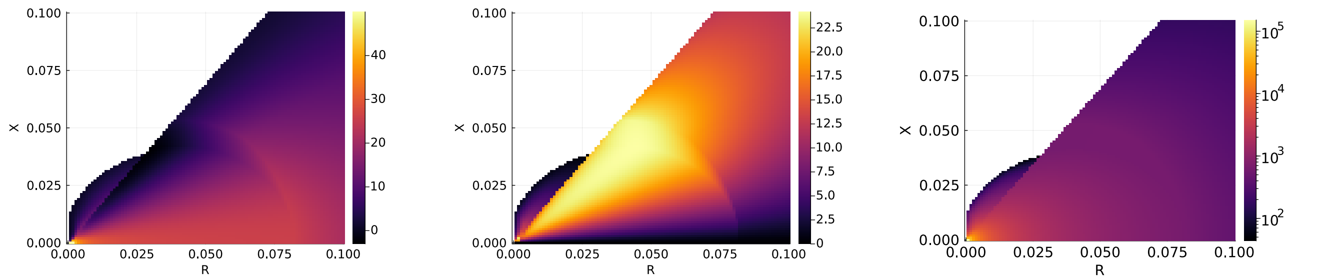
To gain insight into which constraint(s) sometimes prevent the SOC relaxation from obtaining a higher objective value than the nonlinear formulation (i.e., from being inexact), we introduce slack variables in the right side of the equality constraints of (8), i.e., (7b), (7c), and (7e). The problem with slack variables can then be solved over the r and x values for which (8) was infeasible and the relaxation was exact. If non-zero slack appears in only one equation or in multiple equations in such a way that it can be attributed to a single variable, and thus that variable's bounds, then identifying sufficient conditions for an exact relaxation may be straightforward.

The values of the slack variables associated with (7b) and (7e) for the 2-bus network are shown in Fig. 6. There are three distinct regions within the colored portion of the plots. The regions can be described as follows: 1) positive slack in the voltage drop equation and zero slack in all other

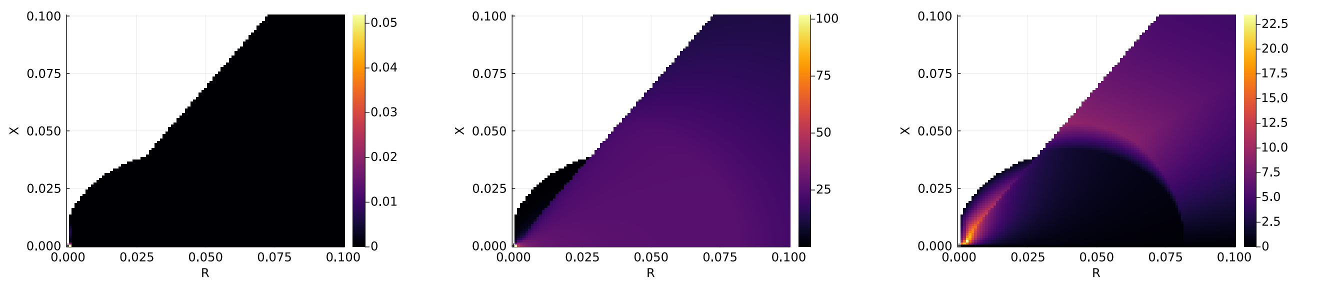


(a) 2-Bus network (b) 56-Bus Network (c) 141-Bus network

Fig. 4. Relationship between r, x and the feasibility (8). Black areas represent values for which (8) is infeasible, i.e., the SOC relaxation is exact. Yellow areas represent values for which (8) is feasible, i.e., the SOC relaxation is not exact. White areas represent values for which (1) was locally infeasible.



(a) δ_{01}^{p} (b) $\delta_{0,1}^{\mathrm{q}}$ (c) δ_{01}^{l}



(d) δ_1^{v} (e) $\delta_1^{\mathrm{p,exp}}$ (f) $\delta_1^{\mathrm{q,exp}}$

Fig. 5. Deviations δ obtained by solving (8) for different r and x values in the 2-bus network. White space indicates values of r and x for which (8) is infeasible, i.e., the SOC relaxation is exact.

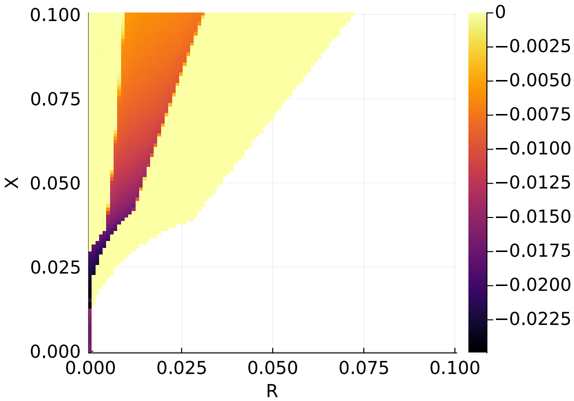
equality constraints, 2) negative slack in the active power balance equation and zero slack in the voltage drop equation, and 3) negative slack in the voltage drop equation and zero slack in all other equality constraints. Similar to the argument used above, the presence of these three distinct regions for this simple 2-bus network suggests that there are multiple sets of conditions that could lead to an exact SOC relaxation.

In summary, for distribution networks that contain lines with both positive resistance and positive reactance, we have shown that there are scenarios under which the SOC relaxation is exact and also scenarios under which the SOC relaxation is not exact for the operating envelope problem. Our numerical analysis, as well as our multiple failed attempts at mathematical exactness and inexactness proofs (omitted for brevity), suggests that there may not be a concise set of conditions for which the relaxation is or is not exact. Given this lack of clear and provable conditions for an exact relaxation and the undesirable consequences of an inexact relaxation in the context of the operating envelope problem, as illustrated in Section III-C and in prior work [9], [25], [26], it is ill-advised to use a SOC relaxation to calculate operating envelopes.

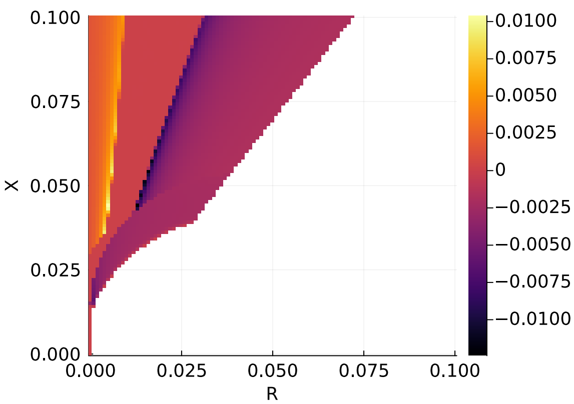
B. Purely Resistive Networks

Results in the previous section for the general case were complicated. However, the results did suggest that it might be possible to derive conditions for purely resistive or purely reactive networks. Therefore, we now consider the case of purely resistive networks, i.e., $\forall\, i'i \in \mathcal{L}$, $r_{i'i} > 0$ and $x_{i'i} = 0$. Again, let $\mathbf{x}^* = (\mathbf{p}^{\exp *}, \mathbf{q}^{\exp *}, \mathbf{p}^*, \mathbf{q}^*, \mathbf{l}^*, \mathbf{v}^*)$ be the optimal solution to (5). Consider a particular bus $i \neq 0$ and assume $v_i^* < \bar{v}$. Now, consider the following candidate solution $\hat{\mathbf{x}}$ to (6):

$$
\begin{aligned}
&\hat{p}_i^{\exp} = p_i^{\exp *} + \frac{\varepsilon}{r_{i'i}}, && \hat{p}_k^{\exp} = p_k^{\exp *},\ \forall k \in \mathcal{N} \setminus \{i\}\,, \\
&\hat{v}_i = v_i^* + \varepsilon, && \hat{v}_k = v_k^*,\ \forall k \in \mathcal{N} \setminus \{i\}\,, \\
&\hat{l}_{i'i} = l_{i'i}^* + \frac{\varepsilon}{r_{i'i}^2}, && \hat{l}_{kn} = l_{k'k}^*,\ \forall k'k \in \mathcal{L} \setminus \{i'i\}\,, \\
&\hat{p}_{k'k} = p_{k'k}^*,\ \forall k'k \in \mathcal{L}, \\
&\hat{q}_{k'k} = q_{k'k}^*,\ \forall k'k \in \mathcal{L}, \\
&\hat{q}_k^{\exp} = q_k^{\exp *},\ \forall k \in \mathcal{N},
\end{aligned}
$$



(a) Slack in active power balance (7b)



(b) Slack in voltage drop (7e)

Fig. 6. Relationship between r, x and the slack variables in (8).

where $0 < \varepsilon \leq \overline{v} - v_i^*$ is a small perturbation. Since $\varepsilon > 0$ and $r_{i'i} > 0 \; \forall \; i'i \in \mathcal{L}$, this candidate solution will have a strictly larger objective value than $\mathbf{x}^*$. We will now show that this candidate solution is feasible for (6) but not feasible for (5), i.e., the SOC relaxation is not exact. To do this, it suffices to show that (1c)–(1f) and (1h)–(1k) hold for $\hat{\mathbf{x}}$ and (2) holds with strict inequality for at least one line.

At bus i, plugging $\hat{\mathbf{x}}$ into (1c) gives

$$p_{i'i}^* - r_{i'i}\left(l_{i'i}^* + \frac{\varepsilon}{r_{i'i}^2}\right) + p_i^{\text{exp}*} + \frac{\varepsilon}{r_{i'i}} = \sum_{j \in \mathcal{C}^i} p_{ij}^*,$$
$$p_{i'i}^* - r_{i'i}l_{i'i}^* - \frac{\varepsilon}{r_{i'i}} + p_i^{\text{exp}*} + \frac{\varepsilon}{r_{i'i}} = \sum_{j \in \mathcal{C}^i} p_{ij}^*,$$
$$p_{i'i}^* - r_{i'i}l_{i'i}^* + p_i^{\text{exp}*} = \sum_{j \in \mathcal{C}^i} p_{ij}^*,$$

which must hold if $\mathbf{x}^*$ is optimal for (5). For any bus $k \neq i$, plugging $\hat{\mathbf{x}}$ into (1c) gives $p_{k'k}^* - r_{k'k}l_{k'k}^* + p_k^{\text{exp}*} = \sum_{n \in \mathcal{C}^k} p_{kn}^*$, which holds by the same argument. Thus, (1c) holds for $\hat{\mathbf{x}}$.

If $x_{k'k} = 0$, then (1d) at any bus k given $\hat{\mathbf{x}}$ is $q_{k'k}^* + q_k^{\text{exp}*} = \sum_{n \in \mathcal{C}^k} q_{kn}^*$. Again, if $\mathbf{x}^*$ is optimal for (5), then this proves that (1d) holds for $\hat{\mathbf{x}}$.

Next, consider (1e) for $\hat{\mathbf{x}}$ at bus i,

$$-\left(p_i^{\text{exp}*} + \frac{\varepsilon}{r_{i'i}}\right)\alpha^{\text{pf}} \leq q_i^{\text{exp}*} \leq \left(p_i^{\text{exp}*} + \frac{\varepsilon}{r_{i'i}}\right)\alpha^{\text{pf}}.$$

Since $\frac{\varepsilon}{r_{i'i}} > 0$, then $-\left(p_i^{\text{exp}*} + \frac{\varepsilon}{r_{i'i}}\right)\alpha^{\text{pf}} < -p_i^{\text{exp}*}\alpha^{\text{pf}}$ and $\left(p_i^{\text{exp}*} + \frac{\varepsilon}{r_{i'i}}\right)\alpha^{\text{pf}} > p_i^{\text{exp}*}\alpha^{\text{pf}}$, showing that the above holds if (1e) holds for $\mathbf{x}^*$. At any bus $k \neq i$, $\hat{p}_k^{\text{exp}} = p_k^{\text{exp}*}$ and $\hat{q}_k^{\text{exp}} = q_k^{\text{exp}*}$ so it is trivial to show that (1e) holds. Thus, (1e) holds at all buses for $\hat{\mathbf{x}}$.

Next, consider (1f) for $\hat{\mathbf{x}}$ at bus i,

$$v_i^* + \varepsilon = v_{i'}^* - 2(r_{i'i}p_{i'i}^*) + r_{i'i}^2\left(l_{i'i}^* + \frac{\varepsilon}{r_{i'i}^2}\right),$$
$$v_i^* + \varepsilon = v_{i'}^* - 2(r_{i'i}p_{i'i}^*) + r_{i'i}^2 l_{i'i}^* + \varepsilon,$$
$$v_i^* = v_{i'}^* - 2(r_{i'i}p_{i'i}^*) + r_{i'i}^2 l_{i'i}^*,$$

which must hold if $\mathbf{x}^*$ is optimal for (5). For any bus $k \neq i$, $\hat{v}_k = v_k^*$, $\hat{p}_{k'k} = p_{k'k}^*$, and $\hat{l}_{k'k} = l_{k'k}^*$ so it is trivial to show that (1f) holds. Thus, (1f) holds at all buses for $\hat{\mathbf{x}}$.

For all lines $k'k \in \mathcal{L}$, $\hat{p}_{k'k} = p_{k'k}^*$ and $\hat{q}_{k'k} = q_{k'k}^*$ so it is trivial to show that (1h) holds for $\hat{\mathbf{x}}$.

Next, consider voltage limits (1i) at bus i: $\underline{v} \leq v_i^* + \varepsilon \leq \overline{v}$. The lower limit holds because $\varepsilon > 0$ and $\underline{v} \leq v_i^*$. Recall that we defined $\varepsilon \leq \overline{v} - v_i^*$, and thus the upper limits holds because $v_i^* + \varepsilon \leq \overline{v}$. For any bus $k \neq i$, $\hat{v}_k = v_k^*$ so it is trivial to show that (1i) holds for $\hat{\mathbf{x}}$. Similarly, since i is not the substation bus then (1j) holds. Since $\varepsilon > 0$, it follows that $l_{k'k}^* + \frac{\varepsilon}{r_{k'k}^2} \geq 0 \; \forall k'k \in \mathcal{L}$, if $\mathbf{x}^*$ is optimal for (5).

Finally, it must be shown that (2) holds with strict inequality on at least one line. Consider the line connecting buses i' and i, and for simplicity, consider (2) in its equivalent form $p_{i'i}^2 + q_{i'i}^2 \leq l_{i'i}v_{i'}$. Plugging in $\hat{\mathbf{x}}$, we get

$$(p_{i'i}^*)^2 + (q_{i'i}^*)^2 \leq \left(l_{i'i}^* + \frac{\varepsilon}{r_{i'i}^2}\right)(v_{i'}^* + \varepsilon),$$
$$(p_{i'i}^*)^2 + (q_{i'i}^*)^2 \leq l_{i'i}^*v_{i'}^* + v_{i'}^*\frac{\varepsilon}{r_{i'i}^2} + l_{i'i}^*\varepsilon + \frac{\varepsilon^2}{r_{i'i}^2},$$
$$(p_{i'i}^*)^2 + (q_{i'i}^*)^2 = l_{i'i}^*v_i^* < l_{i'i}^*v_{i'}^* + v_{i'}^*\frac{\varepsilon}{r_{i'i}^2} + l_{i'i}^*\varepsilon + \frac{\varepsilon^2}{r_{i'i}^2}.$$

The last line follows from the optimality of $\mathbf{x}^*$ for (5) and the fact that $l_{i'i}^* \geq 0$, $v_{i'}^* \geq 0$, $\varepsilon > 0$, and $r_{i'i} > 0 \; \forall \; i'i \in \mathcal{L}$. Thus, we have proven that $\hat{\mathbf{x}}$ is feasible for (6), not feasible for (5), and leads to a higher objective value. This proves that for purely resistive, radial networks, the SOC formulation of the operating envelope problem is not exact if at least one non-substation bus has a voltage strictly lower than the upper voltage limit at optimality of the nonlinear formulation.

With the exception of the 2-bus network, all of the test cases we ran with $r > 0$ and $x = 0$ satisfied the condition that at least one non-substation bus had a voltage strictly lower than the upper voltage limit at optimality of the nonlinear formulation. In the 2-bus network, the only non-substation bus voltage was at its upper limit over the entire range of tested $r > 0$ values, yet the relaxation was still always inexact.

To the best of our knowledge, this is the first proof that identifies conditions under which a relaxation of an OPF problem *fails* to be exact. Prior work in the area of OPF relaxations attempt to prove conditions under which the relaxation is exact. Yet, it is also of value to know under what conditions the relaxation will be sure to fail. The proof that the SOC relaxation will fail to be exact in the operating envelope problem when a network is purely resistive aligns with our initial suspicion that the SOC relaxation is able to achieve larger operating envelopes by nonphysically increasing losses.

C. Purely Reactive Networks

We now consider the case of purely reactive networks, i.e., $\forall\ i'i \in \mathcal{L}$, $r_{i'i} = 0$ and $x_{i'i} > 0$. The following proof was inspired by the proof given in [18]. For the sake of contradiction, assume that $\mathbf{x}^* = (\mathbf{p}^{\exp *}, \mathbf{q}^{\exp *}, \mathbf{p}^*, \mathbf{q}^*, \mathbf{l}^*, \mathbf{v}^*)$ is the optimal solution to (6) such that (2) holds with strict inequality on a particular line connecting buses i' and i, i.e., $(p^*_{i'i})^2 + (q^*_{i'i})^2 < l^*_{i'i} v^*_{i'}$ and (6) is not exact. Now, consider the following candidate solution $\hat{\mathbf{x}}$ to (6):

$$\begin{aligned} &\hat{p}_i^{\exp} = p_i^{\exp *} + \varepsilon, && \hat{p}_k^{\exp} = p_k^{\exp *},\ \forall k \in \mathcal{N} \setminus \{i\}, \\ &\hat{p}_{i'i} = p^*_{i'i} - \varepsilon, && \hat{p}_{k'k} = p^*_{k'k},\ \forall k'k \in \mathcal{L} \setminus \{i'i\}, \\ &\hat{q}_{k'k} = q^*_{k'k},\ \forall k'k \in \mathcal{L}, && \\ &\hat{v}_k = v^*_k,\ \forall k \in \mathcal{N}, && \\ &\hat{l}_{k'k} = l^*_{k'k},\ \forall k'k \in \mathcal{L}, && \\ &\hat{q}_k^{\exp} = q_k^{\exp *},\ \forall k \in \mathcal{N}, && \end{aligned}$$

where $\varepsilon > 0$ is a small perturbation (not necessarily the same perturbation as in the purely resistive case). Since $\varepsilon > 0$, $\hat{\mathbf{x}}$ will have a strictly larger objective value than $\mathbf{x}^*$ if it is feasible, which would contradict the optimality of $\mathbf{x}^*$. To check feasibility, it suffices to check that there exists $\varepsilon > 0$ such that $\hat{\mathbf{x}}$ satisfies (1c)–(1f), (1h)–(1k), and (2). Since $\mathbf{x}^*$ is feasible, (1d), (1f), and (1i)–(1k) hold for $\hat{\mathbf{x}}$ across all buses and lines. Similarly, (1c), (1e), (2), and (1h) hold for $\hat{\mathbf{x}}$ on all lines $k'k \neq i'i$ and at all buses $k \neq i$. We will now show that (1c), (1e), (2), and (1h) hold for $\hat{\mathbf{x}}$ on line $i'i$ and at bus i. Specifically, for (2), we will show that improving the objective value will ultimately shape the solution such that (2) approaches equality.

At bus i, plugging $\hat{\mathbf{x}}$ into (1c) gives

$$p^*_{i'i} - \varepsilon + p_i^{\exp *} + \varepsilon = p^*_{i'i} + p_i^{\exp *} = \sum_{j \in \mathcal{C}^i} p^*_{ij},$$

which must hold if $\mathbf{x}^*$ is feasible for (6).

Next, consider (1e) for $\hat{\mathbf{x}}$ at bus i

$$-\left(p_i^{\exp *} + \varepsilon\right)\alpha^{\mathrm{pf}} \leq q_i^{\exp *} \leq \left(p_i^{\exp *} + \varepsilon\right)\alpha^{\mathrm{pf}}.$$

Since $\varepsilon > 0$, then $-\left(p_i^{\exp *} + \varepsilon\right)\alpha^{\mathrm{pf}} < -p_i^{\exp *}\alpha^{\mathrm{pf}}$ and $\left(p_i^{\exp *} + \varepsilon\right)\alpha^{\mathrm{pf}} > p_i^{\exp *}\alpha^{\mathrm{pf}}$, showing that the above holds if (1e) holds for $\mathbf{x}^*$.

Consider (1h) on line $i'i$ for $\hat{\mathbf{x}}$:

$$\hat{p}^2_{i'i} + \hat{q}^2_{i'i} = (p^*_{i'i} - \varepsilon)^2 + (q^*_{i'i})^2 \leq \bar{s}^2.$$

Thus (1h) only holds for $\hat{\mathbf{x}}$ on line $i'i$ if $(p^*_{i'i})^2 + (q^*_{i'i})^2 \leq \bar{s}^2 + 2\varepsilon p^*_{i'i} - \varepsilon^2$ holds.

For (2) on line $i'i$,

$$\begin{aligned} 0 &> (p^*_{i'i})^2 + (q^*_{i'i})^2 - l^*_{i'i} v^*_{i'} \\ &= (\hat{p}_{i'i} + \varepsilon)^2 + \hat{q}^2_{i'i} - \hat{l}_{i'i}\hat{v}_{i'} \\ &= \hat{p}^2_{i'i} + \hat{q}^2_{i'i} - \hat{l}_{i'i}\hat{v}_{i'} + 2\varepsilon\hat{p}_{i'i} + \varepsilon^2 \end{aligned}$$

Since $r = 0$ and there are no active power losses, then $\hat{p}_{i'i} < 0$ if the sum of net export limits across the buses downstream of bus i is positive, i.e., $\sum_{k \in \mathcal{N}_i^{\mathrm{dn}}} \hat{p}_k^{\exp} > 0$, where $\mathcal{N}_i^{\mathrm{dn}}$ denotes the set of all buses downstream of bus i. If $\hat{p}_{i'i} < 0$, then we can choose any $\varepsilon > 0$ sufficiently small such that $\hat{l}_{i'i} \geq \frac{\hat{p}^2_{i'i} + \hat{q}^2_{i'i}}{\hat{v}_{i'}}$. This proves that the candidate solution is feasible and contradicts the optimality of $\mathbf{x}^*$. This also implies that (2) will hold with equality for the true optimal solution and that the relaxation is exact. We note that because the objective is to maximize $\sum_{k \in \mathcal{N}} \hat{p}_k^{\exp}$, then $\hat{p}_{i'i} < 0$ typically holds. In each test case we ran, active power line flows were all negative when $r = 0$.

The proof that the SOC relaxation will be exact in the operating envelope problem when a network is purely reactive also aligns with our initial intuition. Without resistance, the SOC relaxation is unable to nonphysically increase active power losses and therefore increase the operating envelopes.

V. Conclusions

In this paper we discussed the implications of using a SOC relaxation for calculating operating envelopes. Specifically, we reinforced previous concerns and observations about the inadequacy of operating envelopes calculated using a SOC relaxation in regards to network safety. We also assessed the impact of using our previously proposed objective function modification to tighten the SOC relaxation in the operating envelope problem on a larger network. Results suggest that the modification could be used to define operating envelopes that enforce voltage limits using a SOC relaxation, but that the relaxation may not be exact.

We showed that, despite our intuition to the contrary, there are scenarios in which the SOC relaxation can be exact for the operating envelope problem. With this insight, we attempted to identify the underlying conditions that lead to an exact and an inexact relaxation. For the general and realistic case where the network contains lines with both positive resistance and positive reactance, we were unable to provide simple conditions under which the relaxation will be exact or inexact. However, our numerical analysis suggests that there may be multiple sets of conditions that lead to an exact or an inexact relaxation. For networks with purely resistive lines, we mathematically proved that if there exists at least one bus not at its maximum voltage in the optimal solution of the nonlinear formulation, then SOC relaxation will lead to a larger sum of operating envelopes than the nonlinear formulation (i.e., it will be inexact). For networks with purely reactive lines, we mathematically proved that the SOC relaxation will be exact.

The key takeaway from this work is that given a lack of straightforward conditions for an exact relaxation, it is ill-advised to use a SOC relaxation to determine operating envelopes. Nonlinear formulations, linearizations or inner-approximations are more reliable options for calculating safe operating envelopes.

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