Optimizing Parameters of the LinDistFlow Power Flow Approximation for Distribution Systems

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Abstract—The DistFlow model accurately represents power flows in distribution systems, but the model's nonlinearities result in computational challenges for many applications. Accordingly, a linear approximation known as LinDistFlow (and its three-phase extension LinDist3Flow) is commonly employed. This paper introduces a parameter optimization algorithm for enhancing the accuracy of this approximation for both balanced single-phase equivalent and unbalanced three-phase distribution network models, with the goal of aligning the outputs more closely with those from the nonlinear DistFlow model. Using sensitivity information, our algorithm optimizes the LinDistFlow approximation's coefficient and bias parameters to minimize discrepancies in predictions of voltage magnitudes relative to the nonlinear DistFlow model. The parameter optimization algorithm employs the Truncated Newton Conjugate-Gradient (TNC) method to fine-tune coefficients and bias parameters during an offline training phase to improve the LinDistFlow approximation's accuracy. Numerical results underscore the algorithm's efficacy, showcasing accuracy improvements in L_1 -norm and L_{∞} norm losses of up to 92% and 88%, respectively, relative to the traditional LinDistFlow model. We also assess how the optimized parameters perform under changes in the network topology and demonstrate the optimized LinDistFlow approximation's efficacy in a hosting capacity optimization problem.

Index Terms—DistFlow, LinDistFlow, machine learning, parameter optimization, distribution systems, hosting capacity.

I. INTRODUCTION

Power flow models relating power injections, line flows, and voltages are central to power system design and operation [1]. The AC power flow equations accurately model these relationships. Incorporating the AC power flow equations into optimization problems introduces significant computational challenges due to these equations' nonlinearity, even for the radial networks of typical distribution systems [2]. These challenges are particularly relevant to problems that are run online to inform real-time decisions [3], [4], consider uncertainties [5], and model discrete decisions [6]–[9]. To address these challenges, engineers frequently use power flow approximations that trade accuracy for tractability [10].

Distribution systems are often modeled using an AC power flow formulation known as "DistFlow" [11]–[13]. A linearization of the DistFlow model known as "LinDistFlow" [11]– [13], or one of its variants [14]–[19], is commonly employed to make distribution system optimization problems tractable. An extension known as LinDist3Flow is often used for unbalanced three-phase distribution network models [14]. The traditional LinDistFlow approximation linearizes the DistFlow equations by assuming that the active and reactive line losses are much smaller than the active and reactive line flows. LinDistFlow has been used to site and size capacitors [11], [12], optimize the network topology [13], compute inverter setpoints and perform Volt/VAr control [15], [16], [20], [21], formulate stochastic problems [22], ensure fairness in solar photovoltaic curtailments [23], and calculate electricity prices [24], among many other applications. Despite their extensive applications, the accuracy of such models can vary, particularly outside near-nominal operating regions, as indicated in [16], [25].

Building on ideas from recently developed "adaptive" power flow approximations [26]–[32], this paper proposes an algorithm for optimizing the LinDistFlow parameters to improve its accuracy. Adaptive power flow approximations are linearizations that are tailored to a specific system and operating range of interest. This contrasts with traditional power flow approximations that are often derived using general assumptions about broad classes of systems or are based on a particular nominal operating point. Adaptive power flow approximations invest computing time up front to calculate linearization coefficients in order to achieve increased accuracy when the approximations are deployed in an optimization problem. Thus, adaptive approximations are well suited for settings with both offline and online aspects (e.g., using a dayahead forecast to compute linearization coefficients that are used for online computations in real-time applications [3], [4]) as well as settings where a nonlinear AC power flow model would lead to an intractable formulation [5]–[9].

Our prior work in [33] proposes an adaptive DC power flow approximation that optimizes parameters for a particular system over a specified operating range of interest. Other adaptive power flow approximations minimize worstcase error [26] and expected error [27] or leverage samplebased regression approaches [28]–[30] including techniques for constructing overestimating and underestimating approximations [31]; see [32] for a recent survey.

Contrasting with prior adaptive power flow approximations [26]–[32], this paper develops a parameter optimization algorithm that maintains the structure of the LinDistFlow approximation (and the unbalanced three-phase extension LinDist3Flow) as dictated by the network topology. Thus, the resulting parameter-optimized LinDistFlow approximation has the key advantages of being directly deployable in the many existing distribution system applications that rely on LinDistFlow (e.g., [11]–[13], [15], [16], [20]–[24]) and enabling straightforward modeling of topology changes.

While not using traditional machine learning models like neural networks, our parameter optimization algorithm draws inspiration from methods for training machine learning models. We define a loss function that compares the AC power flow solutions to the approximation's outputs over a set of sampled power injections. With analytically calculated parameter sensitivities, an offline training phase computes the parameter values that minimize this loss function using a Truncated Newton

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Conjugate-Gradient (TNC) method. Using linear programming or mixed-integer linear programming solvers, the optimized parameters are then used in online calculations for real-time settings or in problems for which a nonlinear AC power flow model would lead to intractability. The proposed approach is conceptually similar to our previous work for optimizing the parameters of the DC power flow approximation [33], but is applicable to unbalanced three-phase distribution systems.

Numerical comparisons demonstrate substantial accuracy advantages of our proposed algorithm compared to the traditional LinDistFlow/LinDist3Flow approximation as well as several recent LinDistFlow variants that also tune parameter values [17]–[19]. We also study how the approximation accuracy is affected by changes in network topology. Finally, to illustrate the benefits of our algorithm, we apply the optimized LinDistFlow parameters in a hosting capacity analysis.

To summarize, the key contributions of this paper are:

- Introducing an algorithm that optimizes the LinDistFlow and LinDist3Flow approximations' coefficients and bias parameters to improve their accuracy.
- Training the model across diverse operating conditions while employing the TNC optimization method to extend the algorithm's tractability.
- Conducting extensive evaluations of the proposed parameter optimization algorithm in various loading conditions. These tests demonstrate improved performance over existing linear power flow models.
- Demonstrating the algorithm's capability to compute parameter values that provide improved accuracy across multiple network topologies.
- Illustrating the efficacy of the optimized LinDistFlow approximation in a hosting capacity problem.

The remainder of this paper is organized as follows: Section II reviews the DistFlow model and LinDistFlow approximation. Section III introduces our proposed algorithm. Section IV extends the algorithm to three-phase network models. Section V presents numerical experiments that evaluate the performance of our algorithm. Section VI concludes the paper.

II. POWER FLOW MODELING

This section introduces the DistFlow formulation and its linear approximation, LinDistFlow. We focus on a balanced single-phase equivalent model to present the key concepts and then extend to unbalanced three-phase networks in Section IV.

We first establish notation. Let operators $(\cdot)^{-1}$, $|\cdot|$, $(\cdot)^{\top}$, and $(\cdot)^{-\top}$ denote the square matrix inverse, the absolute value of a number, the transpose of a matrix/vector, and the transpose of a square matrix inverse, respectively. Let $\mathcal{N} := \{0, 1, ..., n\}$ and $\mathcal{E} := \{1, ..., n\}$ denote the sets of buses and lines, respectively, in a distribution network, where $|\mathcal{E}| = |\mathcal{N}| - 1$ for radial networks. Each bus $n \in \mathcal{N}$ has a voltage V_n . Let $\mathcal{N}' = \mathcal{N} \setminus \{0\}$ represent the set of all buses excluding the substation (bus 0). Each bus $n \in \mathcal{N}'$ has a parent (upstream) bus denoted as π_n . Each line $(\pi_n, n) \in \mathcal{E}$ (connection between bus n and its parent π_n) has impedance $z_n = r_n + jx_n$ with resistance r_n and reactance x_n , where $j := \sqrt{-1}$. (Note that lines are identified by their child buses to simplify notation; see Fig. 1.) Let I_n and $S_n = P_n + jQ_n$ denote the current and complex power flows, respectively, on line (π_n, n) . The net power injection at bus n is $s_n =$



Fig. 1. A 2-bus system with line (π_n, n) feeding bus n from its parent π_n .

 $p_n + jq_n$. Additional variables include squared voltage and current magnitudes $v_n = |V_n|^2$ and $\ell_n = |I_n|^2$. $\mathbf{D}_r = \text{diag}(\mathbf{r})$ and $\mathbf{D}_x = \text{diag}(\mathbf{x})$ are diagonal matrices of resistances and reactances, and $\mathbf{p} = [p_1, \dots, p_n]^\top$, $\mathbf{q} = [q_1, \dots, q_n]^\top$, $\mathbf{P} = [P_1, \dots, P_n]^\top$, and $\mathbf{Q} = [Q_1, \dots, Q_n]^\top$ are vectors of active/reactive power injections and flows. Define $\hat{\mathbf{A}} = [\mathbf{a}_0 \mathbf{A}]$ as the $|\mathcal{E}| \times |\mathcal{N}|$ branch-bus incidence matrix describing the connections between the system's buses and branches, where \mathbf{a}_0 is the length- $|\mathcal{E}|$ vector associated with the substation bus and \mathbf{A} is the reduced branch-bus matrix for all buses besides the substation bus. Let $\mathbf{v} := [|V_1|^2, \dots, |V_n|^2]^\top$ represent squared voltage magnitudes and $v_0 = |V_0|^2$ correspond to the substation bus. Bias parameters γ , ρ , and ρ formulated as vectors of length $|\mathcal{N}| - 1$ are optimized using our proposed algorithm to enhance voltage approximation accuracy.

A. DistFlow Model for Single-Phase Equivalent Networks

The DistFlow model [11]–[13] accurately represents voltage, current, and power flow relationships in radial distribution networks. The DistFlow model is:

$$v_n = |V_n|^2, \quad \ell_n = |I_n|^2, \qquad \forall n \in \mathbb{N},$$
(1a)

$$\sum_{k:n \to k} P_k = p_n + P_n - r_n \ell_n, \qquad \forall n \in \mathbb{N},$$
 (1b)

$$\sum_{k:n \to k} Q_k = q_n + Q_n - x_n \ell_n, \qquad \forall n \in \mathbb{N},$$
 (1c)

$$v_n = v_{\pi_n} - 2(r_n P_n + x_n Q_n) + (r_n^2 + x_n^2)\ell_n, \,\forall n \in \mathbb{N}, \,\,(1d)$$
$$v_{\pi_n} \ell_n = P_n^2 + Q_n^2, \qquad \forall n \in \mathbb{N}. \,\,(1e)$$

Equation (1a) defines variables for the squared voltage magnitudes v_n and squared current flow magnitudes ℓ_n which are used in place of the voltage phasors V_n and current phasors I_n . Equations (1b) and (1c) correspond to active and reactive power balance at each bus. Equation (1d) models the change in squared voltage magnitudes across lines. Equation (1e) uses the definition of apparent power to relate the squared voltages v_n , squared currents ℓ_n , and squared apparent power $P_n^2 + Q_n^2$. The DistFlow model is nonlinear due to (1e).

B. Traditional LinDistFlow Approximation

The LinDistFlow approximation linearizes the DistFlow equations by neglecting the line loss terms $r_n\ell_n$ in (1b), $x_n\ell_n$ in (1c), and $(r_n^2 + x_n^2)\ell_n$ in (1d) [11]–[13]. Without these, the nonlinear equation (1e) can be dropped, with the remaining equations linearly relating the squared voltage magnitudes, the power injections, and the line flows:

$$\sum_{k:n \to k} P_k \approx p_n + P_n, \qquad \forall n \in \mathbb{N}, \qquad (2a)$$

$$\sum_{k:n \to k} Q_k \approx q_n + Q_n, \qquad \qquad \forall n \in \mathbb{N}, \qquad (2b)$$

$$v_n \approx v_{\pi_n} - 2(r_n P_n + x_n Q_n), \qquad \forall n \in \mathbb{N}.$$
 (2c)

A matrix form of the LinDistFlow approximation is:

 $\mathbf{D}_r = \operatorname{diag}(\mathbf{r}), \quad \mathbf{D}_x = \operatorname{diag}(\mathbf{x}),$ (3a)

$$\mathbf{p} = \mathbf{A}^{\top} \mathbf{P},\tag{3b}$$

$$\mathbf{q} = \mathbf{A}^{\top} \mathbf{Q},\tag{3c}$$

$$\mathbf{A}\mathbf{v} + v_0\mathbf{a}_0 = 2\mathbf{D}_r\mathbf{P} + 2\mathbf{D}_x\mathbf{Q},\tag{3d}$$

$$\mathbf{v} = v_0 \mathbf{1} + 2\mathbf{A}^{-1} \mathbf{D}_r \mathbf{A}^{-1} \mathbf{p} + 2\mathbf{A}^{-1} \mathbf{D}_x \mathbf{A}^{-1} \mathbf{q}.$$
 (3e)

For radial networks where all lines have positive resistance $(r_n \ge 0)$ and reactance $(x_n \ge 0)$, the LinDistFlow approximation overestimates the voltage magnitudes and underestimates the complex power flows required to supply the loads [34]. The following section presents an algorithm to reduce this approximation error by optimizing the LinDistFlow parameters.

III. OPTIMIZED LINDISTFLOW APPROXIMATION (OLDF)

We first generalize the LinDistFlow approximation by introducing bias parameters γ , ρ , and ϱ that offset the squared voltage magnitudes and active and reactive power injections:

$$\mathbf{v} = v_0 \mathbf{1} + 2\mathbf{A}^{-1} \mathbf{D}_r \mathbf{A}^{-\top} (\mathbf{p} + \boldsymbol{\rho}) + 2\mathbf{A}^{-1} \mathbf{D}_x \mathbf{A}^{-\top} (\mathbf{q} + \boldsymbol{\varrho}) + \boldsymbol{\gamma}.$$
 (4)

Appropriate selection of parameter values for \mathbf{D}_r , \mathbf{D}_x , γ , ρ , and ρ can significantly improve the LinDistFlow approximation's accuracy. We next introduce a machine learning-inspired algorithm which optimizes these parameters to reduce the discrepancy between the voltages output by the LinDistFlow approximation (4) and the DistFlow model (1).

As shown in Fig. 2, we propose a two-phase algorithm for parameter optimization with an *offline* training phase followed by an *online* application phase. The *offline* phase, conducted once, optimizes the values for the parameters \mathbf{D}_r , \mathbf{D}_x , γ , ρ , and ρ . This initial computation seeks to align the LinDistFlow approximation with the DistFlow model's behavior across varied operational scenarios. The optimized parameters are then used in the *online* phase to improve accuracy in various applications while enabling tractable solution with linear programming and mixed-integer linear programming solvers.

To optimize the parameters D_r , D_x , γ , ρ , and ϱ , Section III-A first formulates a loss function that gauges the approximation's accuracy by comparing its voltage predictions to those from the DistFlow model across sampled operational conditions. Section III-B then analytically derives the loss function's sensitivities with respect to the coefficient and bias parameters. Informed by these sensitivities, Section III-C applies the TNC optimization method to minimize the loss function. The resulting optimized parameters are used online for various applications, as benchmarked in Section V.

A. Formulation of the Loss Function

We define a loss function, \mathcal{L} , as the mean square error between the voltage solutions of the DistFlow model (\mathbf{v}_m^{DF}) and



Fig. 2. Flowchart depicting the proposed parameter optimization algorithm.

our optimized LinDistFlow approximation (\mathbf{v}_m^{OLDF}) across a given set of load scenarios $\mathcal{M} = \{1, 2, \dots, S\}$:

$$\mathcal{L}(\mathbf{D}_{r}, \mathbf{D}_{x}, \boldsymbol{\gamma}, \boldsymbol{\rho}, \boldsymbol{\varrho}) = \frac{1}{|\mathcal{M}||\mathcal{N}'|} \sum_{m \in \mathcal{M}} ||\mathbf{v}_{m}^{OLDF} - \mathbf{v}_{m}^{DF}||_{2}^{2},$$
$$= \frac{1}{|\mathcal{M}||\mathcal{N}'|} \sum_{m \in \mathcal{M}} ||v_{0}\mathbf{1} + 2\mathbf{A}^{-1}\mathbf{D}_{r}\mathbf{A}^{-\top}(\mathbf{p} + \boldsymbol{\rho}) + 2\mathbf{A}^{-1}\mathbf{D}_{x}\mathbf{A}^{-\top}(\mathbf{q} + \boldsymbol{\varrho}) + \boldsymbol{\gamma} - \mathbf{v}_{m}^{DF}||_{2}^{2}, \quad (5)$$

where normalization by $\frac{1}{|\mathcal{M}||\mathcal{N}'|}$ adjusts for the system size and number of samples. Using mean square error ensures that larger deviations are more heavily penalized, aligning with typical operational priorities where minimizing the most significant errors is often more important than reducing numerous smaller inaccuracies.

The optimal LinDistFlow parameters minimize \mathcal{L} :

$$\min_{\mathbf{D}_r, \mathbf{D}_x, \boldsymbol{\gamma}, \boldsymbol{\rho}, \boldsymbol{\varrho}} \mathcal{L}(\mathbf{D}_r, \mathbf{D}_x, \boldsymbol{\gamma}, \boldsymbol{\rho}, \boldsymbol{\varrho}).$$
(6)

B. Parameter Sensitivity Analysis

We use the TNC optimization method to solve (6). This method relies on the gradients of the loss function with respect to the parameters, which we present next. We start with the sensitivities for the parameters, represented generically by g^{η} , where η can be any of the parameter sets $\mathbf{D}_r, \mathbf{D}_x, \gamma, \rho$, and ϱ . These sensitivities are calculated as follows:

$$\mathbf{g}^{\eta} = \frac{2}{|\mathcal{M}||\mathcal{N}'|} \sum_{m \in \mathcal{M}} \left. \frac{\partial \mathbf{v}^{OLDF}}{\partial \eta} \right|_{\mathbf{v}_{m}^{OLDF}} \left(\mathbf{v}_{m}^{OLDF} - \mathbf{v}_{m}^{DF} \right), \tag{7a}$$

where $\eta \in {\mathbf{D}_r, \mathbf{D}_x, \gamma, \rho, \varrho}$. The partial derivatives are:

$$\frac{\partial \mathbf{v}^{OLDF}}{\partial \mathbf{D}_r} = 2 \left(\mathbf{A}^{-\top} (\mathbf{p} + \boldsymbol{\rho}) \right)^{\top} \mathbf{A}^{-1}, \tag{7b}$$

$$\frac{\partial \mathbf{v}^{OLDF}}{\partial \mathbf{D}_x} = 2 \Big(\mathbf{A}^{-\top} (\mathbf{q} + \boldsymbol{\varrho}) \Big)^{\top} \mathbf{A}^{-1}, \quad (7c)$$

$$\frac{\partial \mathbf{v}^{OLDF}}{\partial \boldsymbol{\gamma}} = \mathbf{I},\tag{7d}$$

$$\frac{\partial \mathbf{v}^{OLDF}}{\partial \boldsymbol{\rho}} = 2\mathbf{A}^{-1}\mathbf{D}_r\mathbf{A}^{-\top},\tag{7e}$$

$$\frac{\partial \mathbf{v}^{OLDF}}{\partial \boldsymbol{\varrho}} = 2\mathbf{A}^{-1}\mathbf{D}_x\mathbf{A}^{-\top}.$$
 (7f)

Here, I is the identity matrix. These sensitivities enable gradient-based methods such as TNC for optimizing the parameters \mathbf{D}_r , \mathbf{D}_x , γ , ρ , and ρ , as we will describe next.

C. Implementation of the Optimization Solution

The gradients in Section III-B enable the application of gradient-based optimization methods such as TNC [35], [36] to solve the parameter optimization problem (6). The choice of TNC as opposed to other gradient-based optimization methods such as Broyden-Fletcher-Goldfarb-Shanno (BFGS) and limited-memory BFGS [35], [36] is based on the TNC method's superior scalability in our empirical testing.

As illustrated in Algorithm 1, TNC iteratively uses Hessianvector products to approximate the Newton direction, effectively managing memory and computation even in highdimensional spaces. TNC operates by approximating the solution to Newton's equations for a function's local minimum, truncating early to conserve computational resources while still moving significantly towards the minimum. This is particularly advantageous for problems where evaluating the full Hessian matrix is difficult. To compute (6) in order to find the optimal LinDistFlow parameters, we utilize SciPy's scipy.optimize.minimize TNC implementation with the sensitivities from Section III-B, using gradient norm thresholds and iteration limits as termination criteria.

IV. THREE-PHASE LINDISTFLOW (LINDIST3FLOW)

We next discuss extensions to unbalanced three-phase distribution systems [14]. The equivalent of (4) for LinDist3Flow is:

$$\mathbb{V} = \mathbb{V}_0 \mathbf{1} + \mathbb{A}_3^{-1} \mathrm{bdiag}(\mathbb{H}^P) \mathbb{A}_3^{-\top} (\mathbb{P} + \boldsymbol{\rho}_3) + \mathbb{A}_3^{-1} \mathrm{bdiag}(\mathbb{H}^Q) \mathbb{A}_3^{-\top} (\mathbb{Q} + \boldsymbol{\varrho}_3) + \boldsymbol{\gamma}_3.$$
 (8)

where $\mathbb{V} = \begin{bmatrix} v_1^a & v_1^b & v_1^c & \cdots & v_n^a & v_n^b & v_n^c \end{bmatrix}^\top$ is the vector of squared of voltage magnitudes for each phase (a, b, c), \mathbb{A}_3 is the network incidence matrix, $\mathrm{bdiag}(\cdot)$ is the block diagonal operator, $\mathbb{P} = \begin{bmatrix} p_1^a & p_1^b & p_1^c & \cdots & p_n^a & p_n^b & p_n^c \end{bmatrix}^\top$ and $\mathbb{Q} = \begin{bmatrix} q_1^a & q_1^b & q_1^c & \cdots & q_n^a & q_n^b & q_n^c \end{bmatrix}^\top$ are the active and reactive power injection vectors, and the \mathbb{H} matrices for each line $(i, j) \in \mathcal{E}$ are:

$$\mathbb{H}_{ij}^{P} = \begin{bmatrix} -2r_{ij}^{aa} & r_{ij}^{ab} - \sqrt{3}x_{ij}^{ab} & r_{ij}^{ac} + \sqrt{3}x_{ij}^{ac} \\ r_{ij}^{ba} + \sqrt{3}x_{ij}^{ba} & -2r_{ij}^{bb} & r_{ij}^{bc} - \sqrt{3}x_{ij}^{bc} \\ r_{ij}^{ca} - \sqrt{3}x_{ij}^{ca} & r_{ij}^{cb} + \sqrt{3}x_{ij}^{cb} & -2r_{ij}^{cc} \end{bmatrix}$$
(9)

Algorithm 1: Truncated Newton (TNC) Method

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$$\mathbb{H}_{ij}^{Q} = \begin{bmatrix} -2x_{ij}^{aa} & x_{ij}^{ab} + \sqrt{3}r_{ij}^{ab} & x_{ij}^{ac} - \sqrt{3}r_{ij}^{ac} \\ x_{ij}^{ba} - \sqrt{3}r_{ij}^{ba} & -2x_{ij}^{bb} & x_{ij}^{bc} + \sqrt{3}r_{ij}^{bc} \\ x_{ij}^{ca} + \sqrt{3}r_{ij}^{ca} & x_{ij}^{cb} - \sqrt{3}r_{ij}^{cb} & -2x_{ij}^{cc} \end{bmatrix}$$
(10)

Analogous to the single-phase algorithm, we train the parameters \mathbb{H}_{ij}^{P} , \mathbb{H}_{ij}^{Q} , ρ_{3} , ρ_{3} , and γ_{3} for three-phase networks. Parameters ρ_{3} , ρ_{3} , and γ_{3} are the three-phase versions of ρ , ρ , and γ , respectively. To this end, we rewrite the loss function (5) as:

$$\begin{aligned} \mathcal{L}(\mathbb{H}^{P}, \mathbb{H}^{Q}, \boldsymbol{\gamma}_{3}, \boldsymbol{\rho}_{3}, \boldsymbol{\varrho}_{3}) &= \frac{1}{|\mathcal{M}||3\mathcal{N}'|} \sum_{m \in \mathcal{M}} ||\mathbb{V}_{m}^{OLDF} - \mathbb{V}_{m}^{DF}||_{2}^{2} \\ &= \frac{1}{|\mathcal{M}||3\mathcal{N}'|} \sum_{m \in \mathcal{M}} ||\mathbb{V}_{0}\mathbf{1} + \mathbb{A}_{3}^{-1} \mathrm{bdiag}(\mathbb{H}^{P})\mathbb{A}_{3}^{-\top}(\mathbb{P} + \boldsymbol{\rho}_{3}) + \\ & \mathbb{A}_{3}^{-1} \mathrm{bdiag}(\mathbb{H}^{Q})\mathbb{A}_{3}^{-\top}(\mathbb{Q} + \boldsymbol{\varrho}_{3}) + \boldsymbol{\gamma}_{3} - \mathbb{V}_{m}^{DF}||_{2}^{2}. \end{aligned}$$
(11)

The sensitivity of the loss function (11) with respect to the $\mathbb{H}_{ij}^P, \mathbb{H}_{ij}^Q, \rho_3, \rho_3, \rho_3$, and γ_3 parameters, necessary for gradientbased optimization methods such as TNC, can be computed similarly to the single-phase case as in (7). This yields:

$$\mathbf{g}^{\eta} = \frac{2}{|\mathcal{M}||3\mathcal{N}'|} \sum_{m \in \mathcal{M}} \left. \frac{\partial \mathbb{V}^{OLDF}}{\partial \eta} \right|_{\mathbb{V}_{m}^{OLDF}} \left(\mathbb{V}_{m}^{OLDF} - \mathbb{V}_{m}^{DF} \right), \tag{12a}$$

where $\eta \in \{\text{bdiag}(\mathbb{H}^P), \text{bdiag}(\mathbb{H}^Q), \gamma_3, \rho_3, \rho_3\}$, i.e., the parameters to be optimized. The partial derivatives are:

$$\frac{\partial \mathbb{V}^{OLDF}}{\partial \mathrm{bdiag}(\mathbb{H}^P)} = \left(\mathbb{A}_3^{-\top}(\mathbb{P} + \boldsymbol{\rho}_3)\right)^{\top} \otimes \mathbb{A}_3^{-1}, \qquad (12b)$$

$$\frac{\partial \mathbb{V}^{OLDF}}{\partial \mathrm{bdiag}(\mathbb{H}^Q)} = \left(\mathbb{A}_3^{-\top}(\mathbb{Q} + \boldsymbol{\varrho}_3)\right)^{\top} \otimes \mathbb{A}_3^{-1}, \qquad (12c)$$

$$\frac{\partial \mathbb{V}^{OLDF}}{\partial \boldsymbol{\gamma}_3} = \mathbf{I},\tag{12d}$$

$$\frac{\partial \mathbb{V}^{OLDF}}{\partial \boldsymbol{\rho}_3} = \mathbb{A}_3^{-1} \mathrm{bdiag}(\mathbb{H}^P) \mathbb{A}_3^{-\top}, \qquad (12e)$$

$$\frac{\partial \mathbb{V}^{OLDF}}{\partial \boldsymbol{\varrho}_3} = \mathbb{A}_3^{-1} \mathrm{bdiag}(\mathbb{H}^Q) \mathbb{A}_3^{-\top}.$$
 (12f)

Here, \otimes denotes the Kronecker product.

V. NUMERICAL ANALYSIS

This section empirically benchmarks the proposed algorithm "optimized LinDistFlow" (OLDF) against several other related power flow linearizations. Specifically, in addition to the nonlinear DistFlow model (1) that provides the ground truth via an AC power flow solution, we also benchmark our optimized LinDistFlow against the traditional LinDistFlow approximation (LDF) [11]–[13], the parameterized linear power flow (PLPF) approximation from [19], the Lossy DistFlow (LoDF) approximation from [18], and the decoupled linear power flow (DLPF) approximation from [17]. We use various test distribution systems and loading scenarios to replicate the benchmarking methodologies adopted in this literature. These include the balanced single-phase equivalent test cases IEEE 33-bus, IEEE 69-bus, and modified IEEE 123-bus from [37] as well as the 22-bus, 85-bus, 141-bus, case533mt-hi, and case-eu906 systems from MATPOWER [38]. For threephase distribution networks, we use the IEEE 13-bus, IEEE 37-bus, and IEEE 123-bus test cases.

A. Algorithm Training

As in [19], we use 20 power injection scenarios during our algorithm's training phase. These scenarios scale the nominal power injections at each bus by normally distributed multipliers with mean of one and standard deviation of 35%. For the nonlinear DistFlow solutions, we use PowerModels.jl [39] and OpenDSS with the OpenDSSDirect.py package for single- and three-phase networks, respectively, on a PACE computing node at Georgia Tech with a 24-core CPU and 32 GB of RAM. The training algorithm is implemented in Python 3 in a Jupyter Notebook using the TNC method from scipy.optimize.minimize with objective function (5), Jacobian $\mathbf{g} = [\mathbf{g}^{D_r^{\top}}, \mathbf{g}^{D_x^{\top}}, \mathbf{g}^{\gamma^{\top}}, \mathbf{g}^{\rho^{\top}}, \mathbf{g}^{\varrho^{\top}}]^{\top}$, convergence tolerance of 1×10^{-6} per unit, and iteration limit of $100.^1$

B. Performance Metrics

We quantify approximation accuracy by comparing the voltage magnitude outputs (denoted $\mathbf{v}^{[model]}$, where [model] is

OLDF, PLPF, LoDF, LDF, or DLPF for the various approximations) against the nonlinear DistFlow solutions (denoted \mathbf{v}^{DF}). We use max and mean error metrics in per unit (p.u.):

$$\varepsilon_{\max}^{[\text{model}]} = \|\mathbf{v}^{[\text{model}]} - \mathbf{v}^{DF}\|_{\infty}$$
(13)

$$\varepsilon_{\text{avg}}^{[\text{model}]} = \frac{1}{|\mathcal{M}||\mathcal{N}'|} \|\mathbf{v}^{[\text{model}]} - \mathbf{v}^{DF}\|_1$$
(14)

where $|\mathcal{M}|$ is the number of testing samples, $|\mathcal{N}'|$ is the number of non-root nodes in the distribution systems, $\|\cdot\|_{\infty}$ is the L_{∞} norm, and $\|\cdot\|_1$ is the L_1 -norm.

C. Parameter Optimization Analysis

We next present the results of the parameter optimization across various test cases by plotting the parameter values from the traditional LinDistFlow approximation and the LinDistFlow with optimized parameter values.

The box plots in Figs. 3 and 4 show the distributions of \mathbf{D}_r and \mathbf{D}_x parameter values, respectively, for the traditional LinDistFlow and our optimized parameters. Each box plot captures the interquartile range (IQR) with a median line. The whiskers extend to 1.5 times the IQR, with outliers represented as individual points. The horizontal lines at the whiskers' ends indicate the 90th percentile. For each test case, the box plots display two distributions: the LinDistFlow (\mathbf{D}_r^{LDF} or \mathbf{D}_x^{LDF}) and the results from our optimization algorithm (\mathbf{D}_r^{opt} and \mathbf{D}_x^{opt}). The optimized parameter values' distributions align closely with those from existing heuristics for \mathbf{D}_r and \mathbf{D}_x . This indicates that our algorithm yields parameter values in a reasonable range, consistent with conventional heuristics.

Additionly, scatter plots accompanying these box plots compare LinDistFlow (\mathbf{D}_r^{LDF} , \mathbf{D}_x^{LDF}) and optimized (\mathbf{D}_r^{opt} , \mathbf{D}_x^{opt}) parameter values. The red dashed line at 45° in each subplot signifies a one-to-one correlation in the parameter values. These plots also show that the optimized parameters are broadly similar to those from existing heuristics, thus aligning with longstanding power engineering intuition that the line resistances and reactances are key parameters in dictating power flows. Despite the overall consistency with traditional LinDistFlow parameter choices, our results show that the optimized parameters yield significant accuracy improvements.

The box plots in Fig. 5 show the distribution of the bias parameters. Illustrated in Fig. 5a, the optimal values of ρ^{opt} and ϱ^{opt} are notably smaller in magnitude when compared to those of γ^{opt} . Consequently, Fig. 5b presents the ρ^{opt} and ϱ^{opt} parameters separately to highlight their distinct distributions. This disparity suggests simplifying the model by focusing on optimizing only three parameters: \mathbf{D}_r , \mathbf{D}_x , and γ . However, our experimental results indicate that this simplification performs poorly for high-load scenarios. We therefore recommend optimizing all of the parameters.

D. Algorithm Testing

We next characterize the optimized parameter values' performance relative to alternative LinDistFlow formulations under three types of load scenarios: *base*, *high*, and *random* load.

1) Base Load Evaluation: We first assess the LinDistFlow approximation accuracy when using the optimized parameter values on the base loading scenarios in the test cases. Table I shows the OLDF performance metrics, notably the maximum and average voltage magnitude estimation errors (ε_{max} and ε_{avg}), compared to the alternative models PLPF, LoDF, LDF,



Fig. 3. (a) Boxplots showing the distributions of the \mathbf{D}_r parameter values for multiple test cases. Each test case is represented by two boxplots indicating the traditional and optimal \mathbf{D}_r parameter values. (b) Scatter plots comparing the coefficient values \mathbf{D}_r^{LDF} and \mathbf{D}_r^{opt} for various test cases.



Fig. 4. (a) Boxplots showing the distributions of the D_x parameter values for multiple test cases. Each test case is represented by two boxplots indicating the traditional and optimal D_x parameter values. (b) Scatter plots comparing the coefficient values D_x^{LDF} and D_x^{opt} for various test cases.



Fig. 5. Boxplots showing the distributions of the bias parameters for multiple test cases. Each test case is represented by three boxplots indicating the γ^{opt} , ρ^{opt} , and ρ^{opt} parameter values. (b) Plotting only ρ^{opt} and ρ^{opt} parameters for better comparison.



Fig. 6. Voltage profiles for the base case using the nonlinear DistFlow (OpenDSS in black), LinDistFlow with initial parameters (LDF in green), and LinDistFlow with optimized parameters (OLDF in red).

and DLPF. As shown in this table, the OLDF model consistently outperforms its counterparts for all the test cases.

Detailing two examples, Fig. 6 shows the voltage profiles for the 37-bus and the modified 123-bus test cases with the base case loading. While the voltages from the traditional LinDistFlow overestimate the true values, the optimized parameters result in a close alignment with the true DistFlow solution.

2) High Load Evaluation: Following the methodology described in [19], we generated high-load scenarios by scaling the base loads with a factor that ranges from $[-2, -1] \cup [1, 2]$ at a granularity of $\frac{1}{14}$, yielding 30 distinct test scenarios. As shown in Table II, our proposed OLDF algorithm surpasses

the others in reducing the average voltage estimation error (ε_{avg}) across nearly all scenarios with the exception of the 22-bus case. For this case, our OLDF results were better than all except the PLPF approximation where the average error was still quite close (0.00028 p.u. for PLPF vs. 0.00020 p.u. for OLDF). Regarding the maximum errors in the high-load scenarios, no individual approximation consistently dominated the others across all test cases. However, we note that summing the maximum errors across all test cases reveals that the OLDF parameters lead to the best performance in aggregate for this metric. These results show that OLDF parameters trained with scenarios around base-load conditions nevertheless perform well for high-load conditions. Furthermore, as an adaptive

	MODEL EVALUATION - BASE LOAD											
	Test case	ε_{avg}^{LDF}	ε_{avg}^{PLPF}	ε_{avg}^{LoDF}	ε_{avg}^{DLPF}	ε_{avg}^{OLDF}	ε_{\max}^{LDF}	$\varepsilon_{\max}^{PLPF}$	$\varepsilon_{\max}^{LoDF}$	$\varepsilon_{\max}^{DLPF}$	$\varepsilon_{\max}^{OLDF}$	
	22-bus	0.00023	0.00014	0.00236	0.00040	0.00001	0.00030	0.00025	0.00314	0.00066	0.00001	
	33-bus	0.00198	0.00080	0.00288	0.00368	0.00015	0.00284	0.00125	0.00402	0.00638	0.00025	
	69-bus	0.00119	0.00075	0.00112	0.00186	0.00023	0.00388	0.00290	0.00327	0.00766	0.00094	
1	85-bus	0.00531	0.00180	0.00206	0.00942	0.00056	0.00663	0.00221	0.00261	0.01377	0.00075	
	123-bus	0.00218	0.00160	0.00494	0.00348	0.00018	0.00255	0.00186	0.00579	0.00460	0.00034	
	141-bus	0.00152	0.00071	0.00326	0.00280	0.00003	0.00207	0.00099	0.00409	0.00453	0.00004	
	533-bus	0.00031	-	-	-	0.00001	0.00079	-	-	-	0.00005	
	906-bus	0.02552	-	-	-	0.00037	0.02566	-	-	-	0.00119	
30	13-bus	0.02642				0.00901	0.05899				0.03558	
	37-bus	0.02022	-	-	-	0.00325	0.03435	-	-	-	0.00710	
	123-bus	0.01388	-	-	-	0.00340	0.03707	-	-	-	0.01322	
- φ8	13-bus 37-bus 123-bus	0.02022 0.02022 0.01388				$\begin{array}{c} 0.00301\\ \hline 0.00325\\ 0.00340 \end{array}$	0.05899 0.03435 0.03707				$\begin{array}{c} 0.03558\\ 0.00710\\ 0.01322 \end{array}$	

TABLE I

 1ϕ stands for balanced single-phase equivalent networks, and 3ϕ stands for unbalanced three-phase networks.

The best performing method (smallest loss function) is bolded for each test case. All values are in per unit.

IABLE II												
Model Evaluation - High Load												
$\begin{array}{c c c c c c c c c c c c c c c c c c c $											$\varepsilon_{\max}^{OLDF}$	
	22-bus	0.00050	0.00020	0.00376	0.00091	0.00028	0.00132	0.00080	0.00733	0.00280	0.00097	
	33-bus	0.00418	0.00239	0.00556	0.00795	0.00224	0.01573	0.01018	0.01713	0.03133	0.01254	
	69-bus	0.00254	0.00199	0.00278	0.00403	0.00129	0.02253	0.01975	0.01926	0.03929	0.01572	
$ \frac{1}{\phi} $	85-bus	0.01156	0.00817	0.01037	0.02055	0.00719	0.04700	0.03471	0.02487	0.08027	0.03373	
	123-bus	0.00464	0.00301	0.00900	0.00555	0.00258	0.01294	0.00800	0.02017	0.02173	0.00766	
	141-bus	0.00323	0.00234	0.00601	0.00608	0.00182	0.01070	0.00648	0.01501	0.02134	0.00802	
	533-bus	0.00067	-	-	-	0.00037	0.00362	-	-	-	0.00278	
	906-bus	0.02501	-	-	-	0.00358	0.02571	-	-	-	0.00949	
	13-bus	0.06549		-		0.05147	0.18626				0.14138	
30	37-bus	0.06294	-	-	-	0.05907	0.22921	-	-	-	0.20931	
	123-bus	0.03884	-	-	-	0.02379	0.13387	-	-	-	0.12065	

TADLE I

 1ϕ stands for balanced single-phase equivalent networks, and 3ϕ stands for unbalanced three-phase networks. The best performing method (smallest loss function) is bolded for each test case. All values are in per unit.

power flow, the OLDF approximation has the ability to tailor the parameters to perform even better for these conditions by including more training scenarios associated with high loading.

3) Random Load Evaluation: The OLDF approximation's accuracy is further analyzed for random loading conditions using 10,000 scenarios sampled from a uniform distribution of 0% to 150% of the base load. Similar to the base and high load conditions, the performance metrics for the random load conditions detailed in Table III also show the OLDF approximation's dominance, with this approximation having the smallest max and average voltage magnitude errors (ε_{max} and ε_{avg}). Across all test cases, the OLDF accuracy improvement over traditional LDF ranges from 26.42% to 91.67% for average error (ε_{avg}) and from 20.94% to 87.78% for maximum error (ε_{max}). Compared to the best of PLPF and LoDF, OLDF's improvement ranges from 10% to 80% for average error and from 5.56% to 80.70% for max error.

E. Computational Efficiency

As shown in Fig. 2, our algorithm computes LinDistFlow parameters during an offline phase where ample computing time is available. These parameters are then used in online applications where computing time may be limited. Thus, the training process for our proposed algorithm requires computational tractability consistent with an offline context. As shown by the training times in Table IV that range from 0.6451 seconds for the 22-bus case to 25.3917 seconds for the 123-bus case, leveraging mature optimization methods like TNC enables acceptable scalability for the training phase.

The computation times for online uses of the optimized LinDistFlow parameters depend on the particular application for which they are employed. However, since the only changes are to the parameter values and not the mathematical form of the LinDistFlow expressions, online computation times with our optimized parameters should be comparable to existing LinDistFlow approximations. To show this, the average calculation times for the 10,000 random load scenarios in the experiment in Section V-D3 ranged from 0.0004 to 0.1156 seconds, with differences of less than 2% between the LinDistFlow using traditional parameters and our optimized parameters.



Fig. 7. The IEEE 33-bus system with 3 switchable lines and 5 tie-lines.

F. Topology Analysis

Engineers frequently need to both assess the impacts of topology changes for contingency assessments and optimize the topology of distribution systems for voltage management, loss minimization, outage restoration, etc. [40]. Thus, it is important to study the OLDF model's accuracy across topologies.

	MODEL EVALUATION - RANDOM LOAD										
	Test case	ε_{avg}^{LDF}	ε_{avg}^{PLPF}	ε_{avg}^{LoDF}	ε_{avg}^{OLDF}	ε_{\max}^{LDF}	$\varepsilon_{\max}^{PLPF}$	$\varepsilon_{\max}^{LoDF}$	$\varepsilon_{\max}^{OLDF}$		
	22-bus	0.00024	0.00010	0.00220	0.00002	0.00090	0.00057	0.00477	0.00011		
	33-bus	0.00114	0.00051	0.00262	0.00019	0.00443	0.00312	0.00662	0.00124		
	69-bus	0.00077	0.00051	0.00101	0.00017	0.00918	0.00816	0.00476	0.00347		
1ϕ	85-bus	0.00278	0.00056	0.00261	0.00031	0.00735	0.00321	0.00347	0.00257		
	123-bus	0.00120	0.00080	0.00392	0.00072	0.00300	0.00227	0.00624	0.00139		
	141-bus	0.00084	0.00035	0.00273	0.00013	0.00241	0.00108	0.00516	0.00102		
	533-bus	0.00018	-	-	0.00013	0.00097	-	-	0.00061		
	906-bus	0.02548	-	-	0.00051	0.02569	-	-	0.00265		
$\frac{3\phi}{2}$	13-bus	0.02703			0.01335	0.13930			$\overline{0.08872}$		
	37-bus	0.01438	-	-	0.01058	0.07639	-	-	0.05453		
	123-bus	0.01274	-	-	0.00637	0.05455	-	-	0.03506		

TABLE III MODEL EVALUATION - RANDOM LOAD

 1ϕ stands for balanced single-phase equivalent networks, and 3ϕ stands for unbalanced three-phase networks. The best performing method (smallest loss function) is bolded for each test case. All values are in per unit.

TABLE IV	

COMPUTATION TIMES IN SECONDS											
Test	22-	33-	69-	85-	123-	141-	533-	906-	13-	37-	123-
case	bus	bus	bus	bus	bus	bus	bus	bus	bus	bus	bus

 $\begin{smallmatrix} t_{train} 0.6451 & 0.7852 & 4.9879 & 2.7182 & 2.3093 & 2.6942 & 3.4019 & 7.1058 \\ t_{base} & 0.0071 & 0.0092 & 0.0161 & 0.0174 & 0.0112 & 0.0114 & 0.1025 & 0.3061 & 0.0060 & 0.0539 & 1.1495 \\ t_{10000} & 0.0004 & 0.0005 & 0.0008 & 0.0009 & 0.0006 & 0.0014 & 0.0023 & 0.0042 & 0.0182 & 0.0362 & 0.1156 \\ \end{split}$

 t_{train} : Offline computation time to train the parameters.

 t_{base} : Online computation time for the base case loading (i.e., one scenario).

 t_{10000} : Online per-scenario time averaged over 10,000 random scenarios.

TOPOLOGY CHANGES: OPENED AND CLOSED LINES										
Opened	Closed	Opened	Closed	Opened	Closed					
$\begin{array}{c} - \\ (4) \\ (4) \\ (10) \\ (10) \\ (10) \\ (26) \\ (26) \\ (26) \\ (4, 10) \\ (4, 10) \\ (4, 10) \end{array}$	- (33) (35) (37) (34) (35) (36) (36) (37) (33, 34) (33, 35) (33, 36)	$\begin{array}{c} (4, 10) \\ (4, 10) \\ (4, 10) \\ (4, 10) \\ (4, 10) \\ (4, 26) \\ (4, 26) \\ (4, 26) \\ (4, 26) \\ (4, 26) \\ (10, 26) \\ (10, 26) \end{array}$	(34, 35) (34, 37) (35, 36) (35, 37) (36, 37) (33, 36) (33, 36) (35, 37) (35, 36) (35, 37) (36, 37) (36, 37) (34, 35) (34, 36)	$(10, 26) \\(10, 26) \\(10, 26) \\(4, 10, 26) $	(35, 36) (35, 37) (36, 37) (33, 34, 35) (33, 34, 36) (33, 35, 36) (33, 35, 37) (33, 36, 37) (34, 35, 36) (34, 35, 37) (34, 36, 37)					

These are all the valid switching combinations that lead to connected radial configurations for the IEEE 33-bus test case shown in Fig. 7. Note that the table indicates the changes from the configuration shown in Fig. 7.

Addressing topology changes with the LinDistFlow approximation can be approached in various ways. For example, the traditional LinDistFlow approximation allows for straightforward topology adjustments by updating the D_r and \mathbf{D}_x matrices with new values corresponding to the altered topology to calculate voltages. Similarly, in our optimizationbased algorithm, adjustments can be made by excluding optimized parameters for removed lines and incorporating original resistance and reactance values for new lines in \mathbf{D}_{r}^{opt} and \mathbf{D}_x^{opt} , without altering the bias parameters (γ^{opt} , ρ^{opt} , and ρ^{opt}). Successfully maintaining performance with this strategy indicates that our algorithm adapts well to different network topologies, avoiding overfitting to a specific configuration. Alternatively, optimizing parameters specifically for each topology through dedicated optimization could enhance accuracy at the cost of increased computational time and storage for the additional parameters needed. Nevertheless, such calculations could be efficiently executed in parallel for a specified set of topologies, making this process suitable for high-performance computing environments, as each topology's optimization process operates independently from others.



Fig. 8. Heatmap visualization of average error metric (ε_{avg}) across 35 topologies for the IEEE 33-bus test case, with a log-scaled color representation to highlight performance variations. The accompanying vector plot below the heatmap quantitatively assesses the LinDistFlow approximation's baseline performance across the same topologies, facilitating a direct comparison of adaptability and Optimization effectiveness.

To explore these different approaches, we describe a smallscale experiment. Fig. 7 depicts the IEEE 33-bus distribution system, as described in [13], which has 33 nodes and 37 lines. We consider a version of this system with eight switchable lines, specifically, lines 4, 10, and 26 are normally closed switches (NCS) and lines 33 to 37 are tie lines or normally open switches (NOS). With these switchable lines, we can create 35 distinct and valid (i.e., radial and connected) topologies out of 56 possible topologies, as listed in Table V.

We next evaluate the adaptability of optimized parameters, i.e., assess how well parameters optimized for one topology perform in others. To accomplish this, we performed the offline training phase of our proposed algorithm using the same setup as before across the 35 network topologies to obtain a dataset with 35 sets of optimized coefficients and bias parameters. We



Fig. 9. Binary comparison matrix between the traditional LinDistFlow approximation's performance and optimized parameters from our proposed algorithm across 35 topologies in the IEEE 33-bus test case. Each column represents a topology, with green indicating that the optimized parameters for the topology associated with the corresponding row outperform the traditional LinDistFlow approximation and white indicating that the traditional LinDistFlow approximation performance is better.

tested the performance of these optimized parameters using the same 10,000 test samples as before on each topology.

Illustrating this cross-topology assessment, Fig. 8 shows the performance of optimized parameters considering all 35 topologies. Specifically, the heatmap at the top of this figure shows the average error metric, ε_{avg} , of the optimized parameters for a given topology (rows) when applied to different topologies (columns). The horizontal vector plot at the bottom of the figure shows the performance of the traditional LinDistFlow approximation on the same 35 topologies. The heatmap employs a logarithmic scale for color representation to enable comparisons across a broad range of error magnitudes. The results underscore the variability in the algorithm's adaptability, with darker shades indicating lower errors (better performance) and lighter shades denoting higher errors. The vector plot beneath the heatmap contrasts the overall baseline performance of the traditional LinDistFlow approximation.

The matrix in Fig. 9 further illustrates the performance of our optimized parameters relative to the traditional LinDistFlow approximation (i.e., comparing each row from heatmap in Fig. 8 to the horizontal vector plot at the bottom of the figure). This matrix employs a binary color coding—green for topologies where the optimized parameters outperform the traditional LinDistFlow approximation and white where they do not. For example, row 1 in this matrix shows that training the parameters using the base topology results in the optimized parameters outperforming the traditional LinDistFlow approximation on topologies 1, 5, 6, 9, 19, 21, 24, 26, 29, and 31. As another example, training the parameters using topology 2 leads to superior performance on topologies { $[2, 4] \cup 8 \cup$ $[10, 15] \cup [17, 18] \cup 20, \cup [22, 23] \cup 25 \cup 28 \cup 30 \cup 33 \cup 35$ }.

These visualizations show the nuanced performance of the optimized parameters vis-à-vis the traditional LinDistFlow

approximation across topologies. Our future work aims to cluster topologies for which jointly optimized parameter values can provide accurate LinDistFlow approximations.

G. Illustrative Application: Hosting Capacity Computation

As an illustrative example application, this section demonstrates the use of the proposed optimized LinDistFlow model to determine the hosting capacity of inverter-based generation units. Following [41], the hosting capacity problem is:

$$\min_{p_n^g, q_n^g} \sum_{n \in \hat{\mathcal{N}}} \left(\frac{(\overline{p_n^g} - p_n^g)^2}{\overline{p_n^g}} + \xi \frac{q_n^{g^2}}{\overline{s_n^g}} \right)$$
(15a)

s.t.
$$0 \le p_n^g \le \overline{p_n^g}, \quad |q_n^g| \le \sqrt{s_n^{g^2} - p_n^{g^2}}, \quad \forall n \in \hat{\mathbb{N}}$$
 (15b)
 $v_0 = 1, \quad v_n < v_n < \overline{v_n}, \qquad \forall n \in \mathbb{N}'$ (15c)

$$\sqrt{P_n^2 + Q_n^2} \le S_n, \qquad \qquad \forall n : (\pi_n, n) \in \mathcal{E} \quad (15d)$$

$$\left| P_T^2 + Q_T^2 \le S_T, \right|$$
(15e)

$$P_T = \sum_{n:(0,n)\in\mathcal{E}} P_{0n}, \quad Q_T = \sum_{n:(0,n)\in\mathcal{E}} Q_{0n}, \quad (15f)$$

where \hat{N} is the set of buses with inverter-based generators, p_n^g and q_n^g are the active and reactive power generation, $\overline{p_n^g}$ and $\overline{s_n^g}$ are their maximum capacities, S_n and S_T are the upper bounds on the line and transformer flows, and ξ is a weighting factor controlling the tradeoff between active power and reactive power utilization. The constraints ensure adherence to power generation limits (15b) and voltage regulation requirements (15c) along with line (15d) and transformer capacities (15e)–(15f).

An evaluation using the IEEE 33-bus test system illustrates the OLDF model's effectiveness. We set $\overline{s_n^g} = 0.6$ MVA with a 0.98 power factor, $\xi = 0.02$, and voltage limits between 1.05 and 0.95 per unit, with a substation capacity of 10 MVA.

Upon solving (15) with both LDF and OLDF models, we obtain the optimal active and reactive power settings for the inverter-based generation units. By assessing these optimal settings using the original nonlinear DistFlow model, we compare the performance of the OLDF and LDF approximations. Fig. 10 shows that while the traditional LinDistFlow approximation leads to voltage violations at certain buses within the hosting capacity problem (15), the application of the proposed OLDF model avoids such violations.



Fig. 10. Voltage profile of the IEEE 33-bus test case after feeding the active/reactive power injections obtained from solving (15) with LDF (blue curve) and OLDF (orange curve) into the DistFlow equations.

VI. CONCLUSION

The LinDistFlow approximation is often used to improve the computational tractability of optimization problems in distribution systems. This paper introduces a new algorithm that enhances the accuracy of the LinDistFlow approximation for single- and three-phase network models. Inspired by machine learning methods, the algorithm's offline phase optimizes the approximation's coefficients and biases using analytically derived sensitivities within the TNC optimization method. These optimized parameters then provide increased accuracy when used in various applications. Numerical tests demonstrate the algorithm's effectiveness, showing better alignment with nonlinear DistFlow solutions compared to the traditional LinDistFlow and other recent alternatives. Application to a hosting capacity problem further highlights its advantages.

Since our optimized formulation has the same linear form as the traditional LinDistFlow approximation and performs well for varying topologies, the accuracy advantages of the proposed approach can be directly exploited in a wide range of other applications such as those in [11]–[13], [15], [16], [20]–[24]. Our future work aims to explore such applications.

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