Admittance Matrix Concentration Inequalities for Understanding Uncertain Power Networks

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***Abstract*—This paper presents probabilistic bounds for the spectrum of the admittance matrix and classical linear power flow models under uncertain network parameters; for example, probabilistic line contingencies. Our proposed approach imports tools from probability theory, such as concentration inequalities for random matrices with independent entries. It yields error bounds for common approximations of the AC power flow equations under parameter uncertainty, including the DC and LinDistFlow approximations.**

***Index Terms*—Admittance matrix; Concentration inequalities; Sampling; Uncertainty.**

I. Introduction

A. Motivation

In *network problems*, the underlying graphical structure of the electric power system itself may be uncertain, estimated, controlled, or optimized. Such problems are common and challenging in many power engineering settings. In many such cases, a model is partially or completely unknown, resulting in the topology or model parameters being intrinsically uncertain. This can be a source of uncertainty in power system computations, which may cause downstream impacts on decision-making tools.

At the same time, even in the case where the model is known with certainty, many network control problems—such as transmission switching or network reconfiguration—may have vast combinatorial solution spaces, which may be challenging to search through. In both settings, it is desirable to understand how such randomness *propagates through the power flow equations* via the admittance matrix. To this end, we propose *admittance matrix concentration inequalities* as a fundamental tool for working with random power networks models.

B. Novel Contributions

This paper presents probabilistic bounds on spectral perturbations in admittance matrices. Our results make use of the progress in applied probability theory—in particular, sharp matrix concentration inequalities due to [1]—and address several linear power flow models used in the literature for transmission and distribution networks alike, via the linear AC power flow (LACPF) approximation due to [2], [3]. This allows us to model the DC and LinDistFlow approximations under uncertain network parameters.

C. Proposed Approach

We consider a power network modeled by an undirected graph with n nodes and m possible (but not necessarily connected) lines. We reserve the index l for lines (edges), so that $l \in [m] := \{1, 2, \dots, m\}$, and $l = (i, j)$ where $i, j \in [n]$ are the indices reserved for the nodes. We denote by $\boldsymbol{A} \in \{-1, 0, 1\}^{m\times n}$ the branch-to-bus incidence matrix, whose rows $\{\boldsymbol{a}_l\}_{l\in[m]}^{\top}$ are the *incidence vectors* associated with each line $l = (i, j) \in [m]$, where $\boldsymbol{a}_l := \boldsymbol{e}_i - \boldsymbol{e}_j$, with $\boldsymbol{e}_i$ denoting the i-th standard Euclidean basis vector in $\mathbb{R}^n$. We consider a vector $\boldsymbol{w} \in \mathbb{C}^m$ of *random line admittances* $w_l \sim \mathcal{D}_l$, where $\mathcal{D}_l$ is the *admittance distribution* for each line $l = (i, j) \in [m]$, which is not necessarily assumed to be known.

This work studies *random admittance matrices* of the form $\boldsymbol{Y} := \boldsymbol{A}^{\top}\boldsymbol{W}\boldsymbol{A}$, where $\boldsymbol{W} := \mathsf{diag}(\boldsymbol{w})$ is the diagonal matrix with the entries of the complex weight vector $w \in \mathbb{C}^m$ on the diagonal. There are several situations where this model is useful, as discussed in Section I-A. How do such matrices $\boldsymbol{Y}$ behave? We provide precise answers to this question under an array of assumptions; those answers come in the form of upper bounds for the quantities

$$\mathsf{E}\left[\|\boldsymbol{Y}\|\right] \quad \text{and} \quad \mathsf{Pr}\left(\|\boldsymbol{Y}\| \geq t\right) \ \text{ for some } t > 0,$$

i.e., for the expectation and the tail probability of the operator norm $||\boldsymbol{Y}|| = \sqrt{\lambda_{\max}(\boldsymbol{Y}^*\boldsymbol{Y})}$ of the random admittance matrix.

D. Plan of the paper

We outline several applications of the proposed approach, such as bounding the error of linear power flow approximations under parameter uncertainty, including in the I-V formulation of the power flow equations, and bounds for popular linear approximates relative to the AC power flow equations. Additionally, we will demonstrate an application in constructing a screening model for network reconfiguration.

II. Bounding the Spectrum of Admittance Matrices

In this section, we present bounds on the error of the admittance matrix $\boldsymbol{Y} \in \mathbb{C}^{n\times n}$. In particular, we will study the following block Laplacian matrix operator, also known as the *flat start Jacobian*. This matrix captures the spectral behavior of both the admittance matrix, and the behavior of linear approximations of the power flow equations about the flat start; see Appendix A for details.

Definition II.1 (Flat start Jacobian)**.** For an arbitrary power network modeled by an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $n := |\mathcal{N}|$ and $m := |\mathcal{E}|$, define the *linear power flow operator*

$$\boldsymbol{F} := \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{1}$$

where $\boldsymbol{G}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ are graph Laplacian matrices corresponding to networks with sufficiently many strictly positive (i.e., conductance) edge weights such that $\mathsf{rank}(\boldsymbol{G}) \geq n-1$ or real-valued (i.e., susceptance) edge weights, respectively.

The matrix $\boldsymbol{F}$ is the standard power flow Jacobian matrix evaluated at the flat start condition; see Appendix A for a concise derivation. The matrix $\boldsymbol{F}$ is symmetric-indefinite, due to the following practical assumption.

Assumption 1. The conductance and susceptance edges weights are such that

$$\boldsymbol{G} \succeq 0, \qquad \text{and} \qquad \boldsymbol{B} \preceq 0.$$

See [4] for a discussion of the conditions on the susceptances required for $\boldsymbol{B} \preceq 0$ to hold.

In addition to being the flat start Jacobian, the matrix $\boldsymbol{F}$ can also be interpreted as being *equivalent to the admittance matrix up to phase shifts*. By this, we mean that the matrix $\boldsymbol{F}$ is equivalent to the lifted, real-valued version of the standard admittance matrix $\boldsymbol{Y}$. In particular, if we let $\boldsymbol{G} = \mathsf{Re}(\boldsymbol{Y})$ and $\boldsymbol{B} = \mathsf{Im}(\boldsymbol{Y})$, and define

$$\bar{\boldsymbol{Y}} := \begin{bmatrix} \boldsymbol{G} & \boldsymbol{B} \\ \boldsymbol{B} & -\boldsymbol{G} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{2}$$

only the sign of $\boldsymbol{B}$ is flipped in (2) compared to $\boldsymbol{F}$, i.e. $\bar{\boldsymbol{Y}}$ is isomorphic to the Jacobian $\boldsymbol{F}$ under a phase shift of π.

Note that the matrix $\bar{\boldsymbol{Y}}$ is symmetric; it can be easily seen to have the same operator norm as $\boldsymbol{Y}$. Its 2×2 block structure suggests the decomposition as a sum of Kronecker products [5]:

$$\bar{\boldsymbol{Y}} = \sum_{(i,j)\in\mathcal{E}} \begin{bmatrix} g_{ij} & b_{ij} \\ b_{ij} & -g_{ij} \end{bmatrix} \otimes \boldsymbol{E}_{ij} := \sum_{(i,j)\in\mathcal{E}} \boldsymbol{\Upsilon}_{ij} \otimes \boldsymbol{E}_{ij} := \sum_{(i,j)\in\mathcal{E}} \boldsymbol{M}_{ij},$$

where $\{\boldsymbol{\Upsilon}_{ij}\}_{(i,j)\in\mathcal{E}}$ is a sequence of 2×2 *random symmetric matrices* representing the uncertainty in the connection $(i,j) \in \mathcal{E}$, and the sequence of random matrices $\{\boldsymbol{M}_{ij}\}_{ij\in\mathcal{E}}$ decompose the entire network in terms of elementary Laplacian matrices, which we now define.

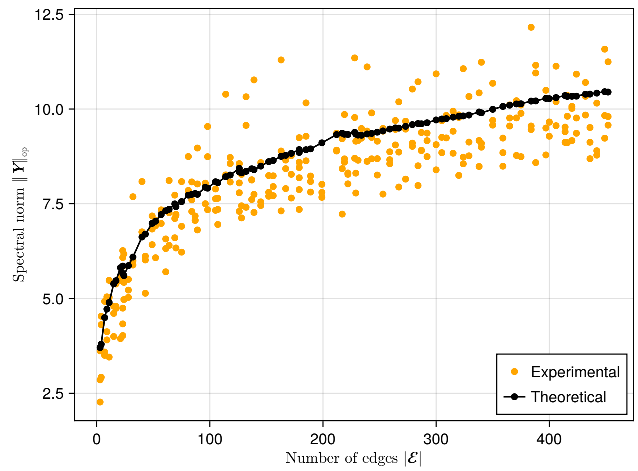

Fig. 1. Comparison between the analytical bound for expected operator norm $\mathsf{E}[||\boldsymbol{Y}||]$ of the admittance matrix and 200 experimental samples, plotted against the number of lines in the network. In this simple numerical experiment, the networks were generated using the homogeneous Erdős-Rényi model, i.e. by switching all possible lines independently with some probability p, and changing p to increase the number of switched lines. Minor discontinuities in the theoretical curve are due to randomness in the number of switched lines.

Definition II.2 (Elementary Laplacian Matrix)**.** For each line $(i,j) \in \mathcal{E}$, let $\boldsymbol{E}_{ij} \succeq 0$ denote the positive-semidefinite *elementary Laplacian matrix*

$$\boldsymbol{E}_{ij} := \boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^\top := (\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top,$$

describing the normalized subgraph between each pair i, j.

A. An illustrative example: Admittances bounded by 1 per-unit

In many applications, it is useful to understand how a power network will behave under uncertainty in the admittance parameters, where the uncertainty has a bounded size. We now give an illustrative example of bounded admittance uncertainty in a network with fixed, known connectivity. In the sequel, we will allow for uncertain connectivity.

Theorem 1 (Concentration of the admittance matrix with fixed connectivity and bounded admittances)**.** *Consider a power system with n nodes and m lines. Let $\Delta = \mathsf{max}_i\, \mathsf{deg}(i)$ be the maximum degree of any node in the network. Suppose that the admittances are distributed according to any distribution $w_l \sim \mathcal{D}$ that satisfies $|w_l| \leq 1$. Then, we have*

$$\mathsf{E}\left[||\boldsymbol{Y}||\right] \leq \sqrt{4\Delta\,\mathsf{log}(4n)} + \frac{2}{3}\,\mathsf{log}(4n).$$

Proof. First, bounding the operator norm uniformly across $\boldsymbol{M}_{ij}$, we have

$$\begin{aligned}||\boldsymbol{M}_{ij}|| &= ||\boldsymbol{\Upsilon}_{ij}|| \cdot ||\boldsymbol{E}_{ij}|| \\ &= 2\sqrt{\lambda_{\max}\left(\boldsymbol{\Upsilon}_{ij}^{\top}\boldsymbol{\Upsilon}_{ij}\right)} \\ &= 2\sqrt{g_{ij}^2 + b_{ij}^2} \\ &\leq 2 \sup_{(i,j)\in\mathcal{E}} |w_{ij}| := L.\end{aligned}$$

On the other hand, denoting with $\boldsymbol{Z}^*$ the conjugate transpose of a complex matrix $\boldsymbol{Z}$, the matrix variance statistic (see [1])

$$\nu\left(\bar{\boldsymbol{Y}}\right) := \| \mathsf{E}[\bar{\boldsymbol{Y}}^2] \|$$

can be expressed as follows:

$$\begin{aligned}\nu\left(\bar{\boldsymbol{Y}}\right) &= \left\|\sum_{(i,j)\in\mathcal{E}} \mathsf{E}\left[\left(\boldsymbol{\Upsilon}_{ij}\otimes\boldsymbol{E}_{ij}\right)\left(\boldsymbol{\Upsilon}_{ij}^{\top}\otimes\boldsymbol{E}_{ij}^{\top}\right)\right]\right\| \\ &\overset{(1)}{=} \left\|\sum_{(i,j)\in\mathcal{E}} \mathsf{E}\left[\left(\boldsymbol{\Upsilon}_{ij}\boldsymbol{\Upsilon}_{ij}^{\top}\right)\otimes\left(\boldsymbol{E}_{ij}\boldsymbol{E}_{ij}^{\top}\right)\right]\right\| \\ &= 2\left\|\sum_{(i,j)\in\mathcal{E}} \begin{bmatrix}\boldsymbol{E}_{ij} & 0 \\ 0 & \boldsymbol{E}_{ij}\end{bmatrix}\right\| \\ &= 2\left\|\begin{bmatrix}\boldsymbol{A}^{\top}\boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^{\top}\boldsymbol{A}\end{bmatrix}\right\| \\ &\overset{(2)}{=} 2\|\boldsymbol{A}^{\top}\boldsymbol{A}\|.\end{aligned} \tag{3}$$

In this chain of equalities, step (1) is by the mixed-product property of the Kronecker product, namely, $(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D}) = (\boldsymbol{AC})\otimes(\boldsymbol{BD})$ for any matrices $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$ with appropriate dimensions; step (2) follows since the operator norm of a block-diagonal matrix is the largest operator norm of any block.

Now, we observe that $\boldsymbol{A}^{\top}\boldsymbol{A}\in\mathbb{R}^{n\times n}$ is the graph Laplacian matrix of the simple undirected graph corresponding to the network topology. we use that $\boldsymbol{Y} = \boldsymbol{D} - \boldsymbol{M}$ where $\boldsymbol{M}$ is the adjacency matrix; since $\boldsymbol{Y}\succeq 0$ and also $\boldsymbol{D}+\boldsymbol{M}\succeq 0$ for the "signless" Laplacian matrix, see e.g. [6], we conclude that

$$-\boldsymbol{D}\preceq\boldsymbol{M}\preceq\boldsymbol{D}, \text{ thus } \|\boldsymbol{M}\|\leq\|\boldsymbol{D}\|$$

and therefore $\|\boldsymbol{Y}\| \leq \|\boldsymbol{D}\| + \|\boldsymbol{M}\| \leq 2\|\boldsymbol{D}\| = 2\Delta$. □

B. Uncertain contingencies

In this section, we present bounds on the spectrum of the admittance matrix under random contingencies, shown in Theorem 2. This bound is useful for analyzing the behavior of the power flow equations under uncertain changes in network topology. This has natural applications in many relevant settings, for example, during natural disasters, public safety power shut-offs, or faults. Throughout this section, we operate under the following assumption.

Assumption 2 (Uncertain contingencies)**.** Suppose that each line $l = (i,j)\in\mathcal{E}$ in a power network is switched closed (resp. switched open) with probability $\mathfrak{p}_l\in[0,1]$ (resp. $1-\mathfrak{p}_l$).

Assumption 2 is equivalent to modeling the power network as an *inhomogeneous* Erdős-Rényi graph. It is highly relevant in the context of risk-based optimal transmission switching; see [7] for example.

We will analyze how the admittance matrix behaves under the setting of Assumption 2. To achieve this, we will define the following notion of *contingency factors*, and the *criticality* of a node.

Definition II.3 (Contingency factors and nodal criticality)**.** Consider a power network in the context of Assumption 2 with line admittances $\{y_l\}$. Define the *contingency factors* $\{c_l\}$ of each line $l = (i,j)\in[m]$ asx

$$c_l := 2\cdot\mathfrak{p}_l\left(1-\mathfrak{p}_l\right)|y_l|^2 \tag{4}$$

and the *degree of criticality* of each node i under the contingency factors $\boldsymbol{c}\in\mathbb{R}_+^m$ is defined as

$$d_i(\boldsymbol{c}) := \sum_{l:l\ni i} c_l = \sum_{l:l\ni i} 2\cdot\mathfrak{p}_l\left(1-\mathfrak{p}_l\right)|y_l|^2. \tag{5}$$

Moreover, we denote the *maximum criticality* under $\boldsymbol{c}$ as

$$\Delta_{\boldsymbol{c}} := \max_{i\,\in\,[n]} d_i(\boldsymbol{c}).$$

In essence, the objects in Definition II.3 are the edge weights, the nodal degrees, and the maximum nodal degree, respectively, of a certain graph Laplacian matrix. In particular, it is the Laplacian matrix that arises from the matrix-valued variance of the admittance matrix under uncertain contingencies, which we define explicitly in the forthcoming result.

Theorem 2 (Concentration with fixed admittances and uncertain contingencies)**.** *Consider a power network in the context of Definition II.3. Let each line $l = (i,j)\in\mathcal{E}$ have admittance $y_{ij}\in\mathbb{C}$ with $|y_{ij}|\leq 1$ per-unit. Define the random line edge weights*

$$w_{ij} := y_{ij}\cdot\xi_{ij}, \quad \xi_{ij}\sim\mathsf{Ber}(\mathfrak{p}_{ij}), \quad (i,j)\in E,$$

and the corresponding random admittance matrix

$$\boldsymbol{Y} := \sum_{(i,j)\in\mathcal{E}} \xi_{ij}y_{ij}\left(\boldsymbol{e}_i-\boldsymbol{e}_j\right)\left(\boldsymbol{e}_i-\boldsymbol{e}_j\right)^{\top} \tag{6}$$

and center as $\tilde{\boldsymbol{Y}} := \boldsymbol{Y} - \mathsf{E}\,\boldsymbol{Y}$. Define the normalized total degree of criticality:

$$\bar{D} := \Delta_{\boldsymbol{c}}^{-1}\sum_{i\in[n]} d_i(\boldsymbol{c}). \tag{7}$$

Then, for all $t\geq\sqrt{2\Delta_{\boldsymbol{c}}}+2/3$, we have

$$\mathsf{Pr}(\|\tilde{\boldsymbol{Y}}\|\geq t) \leq 8\bar{D}\cdot\exp\left(\frac{-t^2}{4\left(\Delta_{\boldsymbol{c}}+t/3\right)}\right); \tag{8}$$

moreover, there exists a constant $C>0$ such that

$$\mathsf{E}\,\|\tilde{\boldsymbol{Y}}\| \leq C\left(\sqrt{2\Delta_{\boldsymbol{c}}\log(1+2\bar{D})} + 2\log(1+2\bar{D})\right). \tag{9}$$

Proof. For each line $l := (i,j) \in \mathcal{E}$, let $\boldsymbol{M}_l = \xi_l y_l \boldsymbol{a}_l \boldsymbol{a}_l^\top$ denote the summand matrices associated with each line in the matrix series (6). Observe that

$$\mathsf{E}\, \boldsymbol{M}_l = \mathfrak{p}_l y_l \boldsymbol{a}_l \boldsymbol{a}_l^\top$$

is the expectation of each elementary admittance matrix. With this, define the *centered* elementary admittance matrices as

$$\tilde{\boldsymbol{M}}_l := \boldsymbol{M}_l - \mathsf{E}\, \boldsymbol{M}_l = (\xi_l - \mathfrak{p}_l)\, y_l \boldsymbol{a}_l \boldsymbol{a}_l^\top.$$

The centered admittance matrix of the network is then

$$\tilde{\boldsymbol{Y}} = \boldsymbol{Y} - \mathsf{E}\, \boldsymbol{Y} = \sum_{l \in E} \tilde{\boldsymbol{M}}_l = \sum_{l \in E} (\xi_l - \mathfrak{p}_l)\, y_l \boldsymbol{a}_l \boldsymbol{a}_l^\top.$$

Naturally, we have that $\mathsf{E}\, \tilde{\boldsymbol{M}}_l = \boldsymbol{0}$ for any l, and $\mathsf{E}\, \tilde{\boldsymbol{Y}} = \boldsymbol{0}$.

Furthermore, for each line l, we have the upper bound

$$\begin{aligned} \left|\left|\tilde{\boldsymbol{M}}_l\right|\right| &= \left|\left|(\xi_l - \mathfrak{p}_l)\, y_l \boldsymbol{a}_l \boldsymbol{a}_l^\top\right|\right| \\ &\overset{(1)}{\leq} |\xi_l - \mathfrak{p}_l| \cdot |y_l| \left|\left|\boldsymbol{a}_l \boldsymbol{a}_l^\top\right|\right|_{\mathsf{op}} \\ &\overset{(2)}{\leq} 2, \end{aligned}$$

where step (1) is by sub-multiplicativity, and step (2) is due to the fact that $\left|\left|\boldsymbol{a}_l \boldsymbol{a}_l^\top\right|\right| = ||\boldsymbol{a}_l||_2^2 = 2$, $|y|_l \leq 1$ and

$$|\xi_l - \mathfrak{p}_l| \leq \max_l |\xi_l - \mathfrak{p}_l| \leq \max_l \{\max\{\mathfrak{p}_l, 1 - \mathfrak{p}_l\}\} \leq 1.$$

Now, we compute the matrix-valued variance of the centered admittance matrix under random contingencies. We have

$$\begin{aligned} \boldsymbol{V} := \mathsf{E}\, \tilde{\boldsymbol{Y}} \tilde{\boldsymbol{Y}}^* &\overset{(1)}{=} \sum_{l \in \mathcal{E}} \mathsf{E}\, \tilde{\boldsymbol{M}}_l \tilde{\boldsymbol{M}}_l^* \\ &= \sum_{l \in \mathcal{E}} \mathsf{E}\left[(\xi_l - \mathfrak{p}_l)^2 |y_l|^2 \boldsymbol{a}_l \boldsymbol{a}_l^\top \boldsymbol{a}_l \boldsymbol{a}_l^\top\right] \\ &\overset{(2)}{=} \sum_{l \in \mathcal{E}} 2\mathfrak{p}_l (1 - \mathfrak{p}_l) |y_l|^2 \boldsymbol{a}_l \boldsymbol{a}_l^\top. \end{aligned}$$

In the above display, step (1) is by independence of the summands, and step (2) is by definition of the Bernoulli variance $\mathsf{E}\,(\xi_l - \mathfrak{p}_l)^2 = \mathfrak{p}_l (1 - \mathfrak{p}_l)$, and $\boldsymbol{a}_l^\top \boldsymbol{a}_l = 2$. Observe that the matrix-valued variance $\boldsymbol{V}$ is itself a graph Laplacian matrix that describes a graph with the same topology as the power network, with the contingency factors as line weights. This Laplacian can be written as

$$\boldsymbol{L} := \boldsymbol{A}^\top \boldsymbol{C} \boldsymbol{A}, \qquad \boldsymbol{C} = \mathsf{diag}(\boldsymbol{c}).$$

Now, we compute the intrinsic dimension of the matrix-valued variance, which is defined as follows.

Definition II.4 (Intrinsic dimension)**.** For any matrix $\boldsymbol{A}$, let

$$\mathsf{intdim}(\boldsymbol{A}) := \frac{\mathsf{tr}(\boldsymbol{A})}{||\boldsymbol{A}||}.$$

First, note that we have

$$\begin{aligned} \mathsf{intdim}(\boldsymbol{V}) &:= \frac{\mathsf{tr}(\boldsymbol{V})}{||\boldsymbol{V}||} \\ &= \frac{2 \sum_{l \in \mathcal{E}} \mathfrak{p}_l (1 - \mathfrak{p}_l) |y_l|^2 \mathsf{tr}\big(\boldsymbol{a}_l \boldsymbol{a}_l^\top\big)}{||\boldsymbol{V}||} \\ &= \frac{4 \sum_{l \in \mathcal{E}} \mathfrak{p}_l (1 - \mathfrak{p}_l) |y_l|^2}{||\boldsymbol{V}||} \end{aligned}$$

The second equality is due to the linearity of the trace, and the third is due to the fact that $\mathsf{tr}\, \boldsymbol{a}_l \boldsymbol{a}_l^\top = 2$. From this juncture, we can now bound the operator norm of the matrix-valued variance as follows. First, note that since $\boldsymbol{V}$ is a Laplacian matrix, it can be written as $\boldsymbol{V} := \boldsymbol{L} = \boldsymbol{D} - \boldsymbol{M}$, where $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ is an adjacency matrix with $M_{ij} = -c_{ij}$ if $(i,j) \in \mathcal{E}$, and zeros along the diagonal, and $D_{ii} := \sum_{l : l \ni i} c_l = d_i(\boldsymbol{c})$. We obtain

$$\begin{aligned} \nu = ||\boldsymbol{V}|| &= ||\boldsymbol{L}|| = ||\boldsymbol{D} - \boldsymbol{M}|| \\ &\overset{(1)}{\leq} \underbrace{||\boldsymbol{D}||}_{=\Delta_{\boldsymbol{c}}} + ||\boldsymbol{M}|| \\ &\overset{(2)}{\leq} \Delta_{\boldsymbol{c}} + \sqrt{||\boldsymbol{M}||_1 \, ||\boldsymbol{M}||_\infty} \\ &\overset{(3)}{=} \Delta_{\boldsymbol{c}} + \sqrt{(\max_j \sum_i |M_{ij}|)(\max_i \sum_j |M_{ij}|)} \\ &= 2\Delta_{\boldsymbol{c}}, \end{aligned}$$

where step (1) is by the triangle inequality and the fact that $\boldsymbol{D}$ is diagonal, and step (2) is by Hölder's inequality, and step (3) is by definition of the matrix norms $||\cdot||_1, ||\cdot||_\infty$. The final equality follows by noting that

$$||\boldsymbol{M}||_1 = ||\boldsymbol{M}||_\infty = ||\boldsymbol{D}|| = \Delta_{\boldsymbol{c}}.$$

Furthermore, we can lower bound the spectral norm by considering the Rayleigh quotient; as $\boldsymbol{V} \succeq \boldsymbol{0}$, we can write $||\boldsymbol{V}||_2 := \mathsf{sup}_{||\boldsymbol{x}|| \leq 1} \boldsymbol{x}^\top \boldsymbol{V} \boldsymbol{x}$. Take $\boldsymbol{x} \leftarrow \boldsymbol{e}_i$, then we always have the lower bound

$$||\boldsymbol{V}|| = \sup_{||\boldsymbol{x}|| \leq 1} \boldsymbol{x}^\top \boldsymbol{V} \boldsymbol{x} \geq \sup_i \boldsymbol{e}_i^\top \boldsymbol{V} \boldsymbol{e}_i = \max_i d_i(\boldsymbol{c}) := \Delta_{\boldsymbol{c}}.$$

Thus,

$$\frac{\sum_i d_i(\boldsymbol{c})}{2\Delta(\boldsymbol{c})} \leq \mathsf{intdim}(\boldsymbol{V}) \leq \frac{\sum_i d_i(\boldsymbol{c})}{\Delta(\boldsymbol{c})} \leq n - 1,$$

where the final inequality is due to the fact that $\mathsf{rank}(\boldsymbol{V}) \leq n - 1$, as $\boldsymbol{V}$ is a Laplacian.

We now prepare to invoke the matrix Bernstein inequality [1]. For non-Hermitian matrices such as (6), the intrinsic dimension factor d is given as

$$d = \frac{2\,\mathsf{tr}(\boldsymbol{V})}{||\boldsymbol{V}||} = 2\,\mathsf{intdim}(\boldsymbol{V}) \leq 2\frac{\sum_i d_i(\boldsymbol{c})}{\Delta(\boldsymbol{c})}.$$

Consequently, for all $t \geq \sqrt{\nu} + L/3 = \sqrt{2\Delta_{\boldsymbol{c}}} + 2/3$, we have

$$\begin{aligned} \mathsf{Pr}(||\tilde{\boldsymbol{Y}}|| \geq t) &\leq 4d \exp\left(\frac{-t^2}{2(\nu + Lt/3)}\right) \\ &\leq 8\left(\frac{\sum_i d_i(\boldsymbol{c})}{\Delta(\boldsymbol{c})}\right) \exp\left(\frac{-t^2}{4(\Delta_{\boldsymbol{c}} + t/3)}\right), \end{aligned}$$

which is the desired result for the tails (8). To yield the expectation bound (9), see [1, Sec. 7.7.4]. Set $\bar{D}$ as in (7). Then, a short calculation reveals

$$\begin{aligned}\mathsf{E}\left[\|\tilde{\boldsymbol{Y}}\|\right] &\leq \sqrt{2\nu\log(1+d)} + \frac{2}{3}L\log(1+d) + 4\sqrt{\nu} + \frac{8}{3}L \\ &\leq C\left(\sqrt{\nu\log(1+d)} + L\log(1+d)\right) \\ &\leq C\left(\sqrt{2\Delta_{\boldsymbol{c}}\log(1+2\bar{D})} + 2\log(1+2\bar{D})\right)\end{aligned}$$

for some universal constant $C > 0$, which is the desired result (13). This completes the proof of Theorem 2. □

Remark. The matrix (6) is not positive-semidefinite, except in extremely restrictive cases; e.g., $\boldsymbol{GB} = \boldsymbol{BG}$ is one sufficient condition. Hence, we must use the Bernstein inequality, as opposed to bounds with potentially simpler forms, like the matrix Chernoff inequality.

III. The Linear Coupled Power Flow Model: Formulation and bounds

The LinDistFlow equations are known to be equivalent to the Linear Coupled Power Flow (LCPF) model in the special case that the network is a tree. We will continue analysis working with the LCPF model for this section as it is more general and results for LinDistFlow follow simply. See Appendix A for more details.

We can decompose $\boldsymbol{F}$, as defined in (17) into the sum of Kronecker products between a particular 2×2 block matrix of admittances and elementary graph Laplacian matrices (Def. II.2), as follows:

$$\begin{aligned}\boldsymbol{F} &= \begin{bmatrix}\boldsymbol{A}^\top\mathsf{diag}(\boldsymbol{g})\boldsymbol{A} & -\boldsymbol{A}^\top\mathsf{diag}(\boldsymbol{b})\boldsymbol{A}\\ -\boldsymbol{A}^\top\mathsf{diag}(\boldsymbol{b})\boldsymbol{A} & -\boldsymbol{A}^\top\mathsf{diag}(\boldsymbol{g})\boldsymbol{A}\end{bmatrix} \\ &= \begin{bmatrix}\sum_{ij\in\mathcal{E}}\boldsymbol{E}_{ij}g_{ij} & -\sum_{ij\in\mathcal{E}}\boldsymbol{E}_{ij}b_{ij}\\ -\sum_{ij\in\mathcal{E}}\boldsymbol{E}_{ij}b_{ij} & -\sum_{ij\in\mathcal{E}}\boldsymbol{E}_{ij}g_{ij}\end{bmatrix} \\ &= \sum_{ij\in\mathcal{E}}\begin{bmatrix}\boldsymbol{E}_{ij}g_{ij} & -\boldsymbol{E}_{ij}b_{ij}\\ -\boldsymbol{E}_{ij}b_{ij} & -\boldsymbol{E}_{ij}g_{ij}\end{bmatrix} \\ &= \sum_{ij\in\mathcal{E}}\underbrace{\begin{bmatrix}g_{ij} & -b_{ij}\\ -b_{ij} & -g_{ij}\end{bmatrix}}_{:=\boldsymbol{\Upsilon}_{ij}}\otimes\boldsymbol{E}_{ij} \\ &= \sum_{ij\in\mathcal{E}}\boldsymbol{\Upsilon}_{ij}\otimes\boldsymbol{E}_{ij} := \sum_{ij\in\mathcal{E}}\boldsymbol{M}_{ij},\end{aligned}$$

where $\boldsymbol{\Upsilon}_{ij}$ is a 2×2 symmetric matrix of admittances for a given line (i,j), as defined above.

A. Spectral properties of elementary power flow Jacobians

1) Boundedness: Let $\boldsymbol{M}_{ij} = \boldsymbol{\Upsilon}_{ij}\otimes\boldsymbol{E}_{ij}$ be the elementary Jacobian corresponding to edge $(i,j)\in\mathcal{E}$. Note that the operator norm of $\boldsymbol{M}_{ij}$ is

$$||\boldsymbol{M}_{ij}|| \overset{(1)}{=} ||\boldsymbol{\Upsilon}_{ij}||\,||\boldsymbol{E}_{ij}|| \overset{(2)}{=} 2\sqrt{g_{ij}^2+b_{ij}^2} = 2\,|y_{ij}|$$

where step (1) is due to the fact that $||\boldsymbol{A}\otimes\boldsymbol{B}|| = ||\boldsymbol{A}||\,||\boldsymbol{B}||$ for any matrices $\boldsymbol{A},\boldsymbol{B}$, and step (2) is due to the fact that $||\boldsymbol{E}_{ij}|| = 2$, and furthermore,

$$\begin{aligned}||\boldsymbol{\Upsilon}_{ij}|| &= \sqrt{\lambda_{\mathsf{max}}\left(\boldsymbol{\Upsilon}_{ij}^\top\boldsymbol{\Upsilon}_{ij}\right)} \\ &= \sqrt{\lambda_{\mathsf{max}}\left(\begin{bmatrix}g_{ij}^2+b_{ij}^2 & 0\\ 0 & g_{ij}^2+b_{ij}^2\end{bmatrix}\right)} \\ &= \sqrt{g_{ij}^2+b_{ij}^2} = |y_{ij}|\,.\end{aligned}$$

Remark. Note that the following identities hold:

$$||\boldsymbol{M}_{ij}|| = \sqrt{2\,\mathsf{tr}\left(\boldsymbol{\Upsilon}_{ij}^\top\boldsymbol{\Upsilon}_{ij}\right)} = \sqrt{2}\,||\boldsymbol{\Upsilon}_{ij}||_F\,, \tag{10}$$

and $||\boldsymbol{M}_{ij}||_F = 2\sqrt{2}\,||\boldsymbol{\Upsilon}_{ij}||\,.$

B. Matrix variance of the LCPF model under uncertainty

In this section, we bound the matrix variance of the LCPF model.

Lemma 1. *Suppose that $\boldsymbol{g},\boldsymbol{b}\in\mathbb{R}^m$ are independent and uniformly distributed on $(m-1)$-dimensional sphere of radius $1/2$; $\boldsymbol{g},\boldsymbol{b}\overset{\mathsf{iid}}{\sim}$ Uniform $\left(\{\boldsymbol{y}\in\mathbb{R}^n\ :\ \boldsymbol{y}^\top\boldsymbol{y}=\frac{1}{2}\}\right)$. Then the matrix-valued variance of the linear power flow operator (Def. II.1) is upper-bounded as*

$$\mathsf{E}\left[\boldsymbol{F}\boldsymbol{F}^*\right] \preceq \frac{2}{n}\mathbf{I}_2\otimes\boldsymbol{A}^\top\boldsymbol{A} := \boldsymbol{V}.$$

Proof. By assumption, each $\boldsymbol{g}\overset{(\mathsf{d})}{=}\boldsymbol{Qz}$, $\boldsymbol{b}\overset{(\mathsf{d})}{=}\boldsymbol{Qz}$, where $\boldsymbol{z}\overset{(\mathsf{iid})}{\sim}$ Normal$(0,\frac{1}{2}\mathbf{I})$ is a vector of iid Gaussians and $\boldsymbol{Q}_g,\boldsymbol{Q}_b\in\mathbb{R}^{n\times n}$ are orthonormal matrices.

First, note that

$$||\boldsymbol{M}_{ij}|| = ||\boldsymbol{\Upsilon}_{ij}||\,||\boldsymbol{E}_{ij}|| \leq 4 := L.$$

The positive semidefinite upper bound for the matrix-valued variance $\mathsf{E}\left[\boldsymbol{F}\boldsymbol{F}^*\right] = \mathsf{E}\left[\boldsymbol{F}^*\boldsymbol{F}\right]$ is then

$$\begin{aligned}\mathsf{E}\left[\boldsymbol{F}\boldsymbol{F}^*\right] &= \sum_{ij\in\mathcal{E}}\mathsf{E}\left[\boldsymbol{M}_{ij}\boldsymbol{M}_{ij}^*\right] \\ &= \sum_{ij\in\mathcal{E}}\mathsf{E}\left[\begin{bmatrix}2\boldsymbol{E}_{ij}g_{ij}^2+2\boldsymbol{E}_{ij}b_{ij}^2 & \mathbf{0}\\ \mathbf{0} & 2\boldsymbol{E}_{ij}b_{ij}^2+2\boldsymbol{E}_{ij}g_{ij}^2\end{bmatrix}\right] \\ &= 2\sum_{ij\in\mathcal{E}}\begin{bmatrix}\boldsymbol{E}_{ij}\left(\mathsf{E}\left[g_{ij}^2+b_{ij}^2\right]\right) & \mathbf{0}\\ \mathbf{0} & \boldsymbol{E}_{ij}\left(\mathsf{E}\left[g_{ij}^2+b_{ij}^2\right]\right)\end{bmatrix} \\ &\overset{(1)}{\preceq} \frac{2}{n}\sum_{ij\in\mathcal{E}}\begin{bmatrix}\boldsymbol{E}_{ij} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{E}_{ij}\end{bmatrix} \\ &\overset{(2)}{=} \frac{2}{n}\begin{bmatrix}\boldsymbol{A}^\top\boldsymbol{A} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{A}^\top\boldsymbol{A}\end{bmatrix} \\ &= \frac{2}{n}\mathbf{I}_2\otimes\boldsymbol{A}^\top\boldsymbol{A},\end{aligned}$$

where step (1) stems from the fact that $\mathsf{E}\left[g_{ij}^2\right]\leq 1$ and $\mathsf{E}\left[b_{ij}^2\right]\leq 1$ by assumption of the uniform distribution over the unit sphere. □

1) Matrix variance statistic: The matrix variance statistic is then given as

$$\begin{aligned}\nu = ||\boldsymbol{V}|| &\overset{(1)}{=} \frac{2}{n}\left|\left|\boldsymbol{A}^\top\boldsymbol{A}\right|\right| \\ &\overset{(2)}{\leq} \frac{2}{n}\left|\left|\boldsymbol{A}^\top\boldsymbol{A}\right|\right|_F \\ &\overset{(3)}{\leq} \frac{2}{n}\sqrt{\mathsf{tr}\left(\boldsymbol{A}^\top\boldsymbol{A}\right)} \\ &\overset{(4)}{\leq} \frac{2}{n}\sqrt{n(n-1)} \\ &\overset{(5)}{\leq} 2.\end{aligned}$$

where step (1) follows from the properties of the Kronecker product, step (2) is due to the fact that $||\cdot|| \leq ||\cdot||_F$,

Theorem 3 (A matrix Bernstein bound [8])**.** *Suppose that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k \in \mathbb{R}^{n\times n}$ are independent, symmetric, zero-mean random matrices such that $||\boldsymbol{X}_i|| \leq R$ always. Let $\boldsymbol{X} := \sum_{i=1}^k \boldsymbol{X}_i$ and define*

$$\nu := ||\mathsf{var}(\boldsymbol{X})|| = \left|\left|\sum_{i=1}^k \mathsf{E}\left[\boldsymbol{X}_i^2\right]\right|\right|.$$

Then, for any $t > 0$, we have

$$\mathsf{Pr}\left(||\boldsymbol{X}|| \geq t\right) \leq 2n \,\mathsf{exp}\left(\frac{-t^2}{2Rt + 4\nu}\right)$$

Theorem 4 (Spectral error of LCPF model under uncertain admittance parameters)**.** *Suppose that the conductance and susceptance parameters of an electric power system model are uncertain, and can be modeled for each line $(i,j) \in \mathcal{E}$ as*

$$g_{ij} = g_{ij}^\bullet + \Delta_{ij}^g \tag{11a}$$
$$b_{ij} = b_{ij}^\bullet + \Delta_{ij}^b, \tag{11b}$$

where $g_{ij}^\bullet, b_{ij}^\bullet \in \mathbb{R}$ are the true parameters, and Δ_{ij}^g, Δ_{ij}^b are bounded uncertainty random variables such that

$$\mathsf{max}\left\{\left|\left|\Delta_{ij}^b\right|\right|, \left|\left|\Delta_{ij}^g\right|\right|\right\} \leq \Delta \quad \forall (i,j) \in \mathcal{E}.$$

Then, for any $t \geq 0$, we have that

$$\mathsf{Pr}\left(||\boldsymbol{F} - \mathsf{E}[\boldsymbol{F}]|| \geq t\right) \lesssim n \,\mathsf{exp}\left(\frac{-t^2}{4\left(\Delta^2 n + \Delta t/3\right)}\right), \tag{12}$$

and

$$\mathsf{E}\left[||\boldsymbol{F} - \mathsf{E}[\boldsymbol{F}]||\right] \leq 2\Delta\sqrt{2}\left(\sqrt{n \log(4n)} + \frac{1}{3}\log(4n)\right) \tag{13}$$

Proof. The positive semidefinite upper bound for the matrix-valued variance $\mathsf{E}\left[\boldsymbol{F}\boldsymbol{F}^*\right] = \mathsf{E}\left[\boldsymbol{F}^*\boldsymbol{F}\right]$ is then

$$\begin{aligned}\mathsf{E}\left[\boldsymbol{F}\boldsymbol{F}^*\right] &= \sum_{ij\in\mathcal{E}} \mathsf{E}\left[\boldsymbol{M}_{ij}\boldsymbol{M}_{ij}^*\right] \\ &= 2\sum_{ij\in\mathcal{E}} \begin{bmatrix}\boldsymbol{E}_{ij}\left(\mathsf{E}\left[g_{ij}^2 + b_{ij}^2\right]\right) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E}_{ij}\left(\mathsf{E}\left[g_{ij}^2 + b_{ij}^2\right]\right)\end{bmatrix} \\ &\overset{(1)}{\preceq} 4\Delta^2 \sum_{ij\in\mathcal{E}}\begin{bmatrix}\boldsymbol{E}_{ij} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E}_{ij}\end{bmatrix} \\ &\overset{(2)}{=} 4\Delta^2\begin{bmatrix}\boldsymbol{A}^\top\boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^\top\boldsymbol{A}\end{bmatrix} \\ &= 4\Delta^2 \mathbf{I}_2 \otimes \boldsymbol{A}^\top\boldsymbol{A},\end{aligned}$$

where step (1) stems from the fact that $\mathsf{E}\left[g_{ij}^2\right] \leq \Delta^2$ and $\mathsf{E}\left[b_{ij}^2\right] \leq \Delta^2$ by assumption of bounded model uncertainty. Then, the matrix variance statistic is bounded as

$$\nu \leq 4\Delta^2\left|\left|\boldsymbol{A}^\top\boldsymbol{A}\right|\right| \leq 4\Delta^2 n.$$

Directly the matrix Bernstein inequality completes the proof. □

IV. Application: Bounding the Error of a Family of Power Flow Linearizations

In this section, we provide an error bound of linear approximations of the AC power flow equations under uncertain admittances. This serves as a useful primitive for further applications on the evaluation of the quality of DC power flow in practical problems such as contingency analysis and network reconfiguration. Following the very recent results of [9], we can utilize the perspective that the power flow equations admit a manifold interpretation to perform such analysis under uncertain admittances. Recounting the setup of [9], let the AC power flow manifold be

$$M_{PF} = \mathrm{gph}(\Phi) = \left\{(x,p,q) \in \mathbb{R}^{4n} : (p,q) = \Phi(x)\right\},$$

with $x = (v,\theta) \in \mathbb{R}^{2n}$, $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, where graph is defined as

$$\mathrm{gph}(f) := \{(x, f(x)) \in \mathbb{R}^{n+m} \mid x \in X\}$$

where $f : X \to Y$, $X \subset \mathbb{R}^n$, and $Y \subset \mathbb{R}^m$. Equivalently, $M_{PF} = F^{-1}(0)$ with $F(x,p,q) = \Phi(x) - (p,q)$ and

$$\mathsf{D}F(x,p,q) = \begin{bmatrix}\mathsf{D}\Phi_x & -I_{2n}\end{bmatrix}$$

is surjective everywhere. Fix a feasible base point

$$\boldsymbol{z}_\star = (\boldsymbol{x}_\star, \boldsymbol{p}_\star, \boldsymbol{q}_\star) = (\boldsymbol{x}_\star, \Phi(\boldsymbol{x}_\star)) \in M_{PF}.$$

For a step $h \in \mathbb{R}^{2n}$ tangent at z_* (i.e. implicit function theorem evaluated at z_* s.t. $(h, \mathsf{D}\Phi_{x_*}h) \in T_{z_*}M_{PF}$), form the tangent point $\bar{z} := z_* + (h, \mathsf{D}\Phi_{x_*}h)$, where $T_z M_{PF} = \mathsf{ker}\begin{bmatrix}\mathsf{D}\Phi_x & -I_{2n}\end{bmatrix}$ is the tangent space at z.

Proposition 1. *Let $\bar{z} \in T_z M_{PF}$ be a point in the linear tangent space of the power flow manifold about $z \in M_{PF}$. For a random admittance matrix $\boldsymbol{Y}$, defined in a similar manner* *as Theorem 1, the expected Euclidean distance (in its ambient space) of a tangent step from a random AC PF manifold is*

$$\begin{aligned}\mathsf{E}\left[\operatorname{dist}(\bar{z}, M_{PF})\right] &\leq 3\|h\|_\infty \|h\|_2 \,\mathsf{E}\left[||\boldsymbol{Y}||\right] \\ &\leq 3\|h\|_2^2 \left(\sqrt{4\Delta \log(4n)} + \frac{2}{3}\log(4n)\right).\end{aligned}$$

Proof. From [9, Prop III.1], we immediately get

$$\operatorname{dist}(\bar{z}, M_{PF}) \leq 3\,||F(\bar{z})||\,.$$

By definition, $F(x,p,q) = \Phi(x) - (p,q)$. Therefore,

$$F(\bar{z}) = \Phi(x_* + h) - (\Phi(x_*) + \mathsf{D}\Phi_{x_*} h) \tag{14}$$

The AC PF manifold can be equivalently represented as

$$M_{PF} = \{(u,s) \in \mathbb{C}^n \times \mathbb{C}^n : s = \Psi_Y(u) := \mathsf{diag}(u)\underline{Yu}\}$$

where $\underline{(\cdot)}$ denotes complex conjugate. Let (u_*, s_*) be the complex form of z_*, so the complex tangent point is

$$(\bar{u}, \bar{s}) := (u_* + h_u, s + \mathsf{D}\Psi_Y(u_*)[h_u])$$

where $h_u \in \mathbb{C}^n$ is the complex voltage step. Since Ψ_Y is quadratic in u,

$$\Psi_Y(u_* + h_u) = \Psi_Y(u_*) + \mathsf{D}\Psi_Y(u_*)[h_u] + \frac{1}{2}\mathsf{D}^2\Psi_Y(u_*)[h_u, h_u]$$

with first and second Fréchet derivatives as

$$\begin{aligned}\mathsf{D}\Psi_Y(u)[h_u] &= \mathsf{diag}(h_u)\underline{\boldsymbol{Y}u} + \mathsf{diag}(u)\underline{\boldsymbol{Y}h_u}, \\ \mathsf{D}^2\Psi_Y[h_u, k_u] &= \mathsf{diag}(h_u)\underline{\boldsymbol{Y}k_u} + \mathsf{diag}(k_u)\underline{\boldsymbol{Y}h_u}.\end{aligned}$$

Subtracting the linearization,

$$\begin{aligned}\Psi_Y(u_* + h_u) - (\Psi(u_*) + \mathsf{D}\Psi_{x_*}[h_u]) &= \frac{1}{2}\mathsf{D}^2\Psi_Y(u_*)[h_u, h_u] \\ &= \mathsf{diag}(h_u)\underline{\boldsymbol{Y}h_u}\end{aligned}$$

Since this is the same form as (14) and since $\mathbb{C}^n \cong \mathbb{R}^{2n}$,

$$\|F(\bar{z})\|_2 = \|\,\mathsf{diag}(h_u)\underline{\boldsymbol{Y}h_u}\|_2$$

and by standard norm inequalities, we have

$$\begin{aligned}\|F(\bar{z})\|_2 &= \|\,\mathsf{diag}(h_u)\underline{\boldsymbol{Y}h_u}\|_2 \\ &\leq ||\mathsf{diag}(h)||\,||\boldsymbol{Y}||\,\|h\|_2 = \|h\|_\infty\,||\boldsymbol{Y}||\,\|h\|_2 \\ &\leq \|h\|_2^2\,||\boldsymbol{Y}||\,.\end{aligned}$$

Plugging into [9, Prop III.1]

$$\begin{aligned}\operatorname{dist}(\bar{z}, M_{PF}) &\leq 3\|F(\bar{z})\| \\ &\leq 3\|h\|_\infty\,||\boldsymbol{Y}||\,\|h\|_2 \\ &\leq 3\|h\|_2^2\,||\boldsymbol{Y}||\end{aligned}$$

Taking expectations on both sides and using Thm 1 yields

$$\begin{aligned}\mathsf{E}\left[\operatorname{dist}(\bar{z}, M_{PF})\right] &\leq 3\|h\|_\infty \|h\|_2 \,\mathsf{E}\left[||\boldsymbol{Y}||\right] \\ &\leq 3\|h\|_2^2 \left(\sqrt{4\Delta \log(4n)} + \frac{2}{3}\log(4n)\right).\end{aligned}$$

□

Proposition 2. *Suppose the special case where we assume a lossless network (i.e. $\boldsymbol{Y} = -j\boldsymbol{B}$) with the susceptances are random variables of the form $w_e = b_e \cdot s_e$ where b_e is the physical susceptance and $s_e \sim \mathsf{Ber}(p_e)$, where p_e is the rate at which line e is switched closed. From some constant $C > 0$, the expected distance from a local point on the linear tangent space to the random AC PF manifold is*

$$\begin{aligned}\mathsf{E}\left[\operatorname{dist}(\bar{z}, M_{PF})\right] &\leq 3C\|h\|_\infty \|h\|_2 \\ &\left(\sqrt{2\Delta_{\boldsymbol{c}} \log(1 + 2\bar{D})} + 2\log(1 + 2\bar{D})\right)\end{aligned}$$

Proof. Following directly from Prop 1, we know that

$$\mathsf{E}\left[\operatorname{dist}(\bar{z}, M_{PF})\right] \leq 3\|h\|_\infty \|h\|_2 \,\mathsf{E}\left[||\boldsymbol{Y}||\right]$$

so what's left is to bound this special case of $\mathsf{E}\left[||\boldsymbol{Y}||\right]$. By lossless assumption and norm properties,

$$||\boldsymbol{Y}|| = ||-j\boldsymbol{B}|| = ||\boldsymbol{B}||\,.$$

From Theorem 2, we directly get that

$$\mathsf{E}\,\|\boldsymbol{B}\| \leq C\left(\sqrt{2\Delta_{\boldsymbol{c}} \log(1 + 2\bar{D})} + 2\log(1 + 2\bar{D})\right),$$

so the expected distance from the ACPF manifold is

$$\begin{aligned}\mathsf{E}\left[\operatorname{dist}(\bar{z}, M_{PF})\right] &\leq 3C\|h\|_\infty \|h\|_2 \\ &\left(\sqrt{2\Delta_{\boldsymbol{c}} \log(1 + 2\bar{D})} + 2\log(1 + 2\bar{D})\right)\end{aligned}$$

for some constant $C > 0$. □

V. Conclusions

In this paper, we presented a number of results applying matrix concentration inequalities to characterize behaviors of the power flow equations under uncertain admittances. We first derive an expectation bound on the spectrum of the admittance matrix under general distribution assumptions that scales on the network's maximum degree, and use it to develop refined tail bounds of uncertain contingencies expressed through contingency factors and nodal criticality. We then lift these results to the linear coupled power flow (LCPF) operators and show how the induced spectral uncertainty to linear approximations of the AC power flow manifold, producing explicit error bounds for a family of linear power flow models such as DC power flow.

More precisely, our results imply that the expected operator norm of a random admittance matrix under general distributions of bounded admittances grows like $O(\sqrt{\Delta \log n} + \log n)$, where Δ is the maximum degree and n is the numbers of nodes. This quantity has the interpretation that the under light distributional assumptions on the admittances, the effective resistance of a power network concentrates. From a modeling standpoint, controlling spectral uncertainty gives us access to provide guarantees of models (such as linearizations) that related to uncertainty in topology/network parameters which are quantities we care about. Additionally, there are further applications these results support and leave for future work.

A. Applications and Future Work

1) Contingency Analysis: With a suitable Bernoulli parameter on the admittances, analyzing the distribution of power networks subsumes all possible $n-1$ configurations of the network. In particular, one can show that the simplex of all $n-1$ contingencies does not violate any line flow constraint. For example, in the context of the DC approximation, the results of the present paper can show that

$$\mathsf{Pr}(||\boldsymbol{f}||_\infty > \epsilon) \leq \delta(\epsilon),$$

for an appropriate choice of (ϵ, δ). Here,

$$\boldsymbol{f} = \boldsymbol{S}\boldsymbol{W}\boldsymbol{A}\boldsymbol{\theta}, \qquad \boldsymbol{s} \in \Delta^n_{n-1}$$

with $\boldsymbol{S} := \mathsf{diag}(\boldsymbol{s})$ and $\boldsymbol{W} = \mathsf{diag}(\boldsymbol{w})$ is a random vector of approximate line flows with Δ^n_{n-1} denotes the set of all $n-1$ contingency switching vectors.

2) Network Reconfiguration: Network reconfiguration is of great interest in recent power systems research, particularly for modern applications, such as in congestion management and grid planning. The combinatorial solution space of such problems implies that analyzing or sampling from a family of possible network configurations is a potentially relevant task. The theory of the present paper is directly applicable to such a setting, as it describes the behavior of such non-deterministic power flow models.

3) Evaluations of Linearizations in Specialized Problems: Following up on the results on the error bounds developed in this paper, the evaluation of linearizations such as DC power flow in more specific problems such as contingency analysis and network reconfiguration are of great interest to practitioners. This probabilistic framework could allow screening of linearization choices conformal to specific parameters of the problem. Moreover, applications involving bounding the error of locational marginal prices (LMPs) are also tractable, leveraging results that connect admittance matrices to LMP sensitivities [10], [11].

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Appendix

A. A fresh derivation of the Linear Coupled Power Flow Model (LCPF)

Theorem 5 (Linear Coupled Power Flow Model [2], [12], [3])**.** *Consider the flat start condition $\boldsymbol{u}_\star := \mathbf{1} + \mathrm{j}\mathbf{0}$, and suppose that $\boldsymbol{\omega} = \mathbf{0} + \mathrm{j}\mathbf{0}$. Then, the linear coupled power flow manifold around $\boldsymbol{u}_\star$ is the linear space*

$$\mathcal{M}_\star := \left\{ \boldsymbol{x} \in \mathbb{R}^{4n} : \boldsymbol{F}(\boldsymbol{x}_\star)(\boldsymbol{x} - \boldsymbol{x}_\star) = \mathbf{0}_{2n} \right\}, \qquad (15)$$

where $\boldsymbol{F} : \mathbb{R}^{4n} \to \mathbb{R}^{2n \times 4n}$ is the Jacobian of $\mathcal{F}$ at the nominal state $\boldsymbol{x}_\star$, which we write as

$$\boldsymbol{F}(\boldsymbol{x}_\star) = \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \\ -\boldsymbol{B} & -\boldsymbol{G} & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M} & -\mathbf{I}_{2n\times 2n} \end{bmatrix}. \qquad (16)$$

We define the $2n \times 2n$ matrix $\boldsymbol{M}$ as the linear power flow matrix*. This matrix defines the linear power flow model*

$$\boxed{\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\theta} \end{bmatrix}} \qquad (17)$$

where $\boldsymbol{\epsilon} := \boldsymbol{v} - \mathbf{1}$. If the network is a tree with n non-reference nodes and n edges, the inverse of the linear power flow matrix $\boldsymbol{M}$ is given in closed form as

$$\boldsymbol{M}^{-1} := \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{X} \\ \boldsymbol{X} & -\boldsymbol{R} \end{bmatrix}, \qquad (18)$$

where $\boldsymbol{R}, \boldsymbol{X} \succ 0$ are resistance and reactance matrices. Thus,

$$\boxed{\begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{X} \\ \boldsymbol{X} & -\boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix},} \qquad (19)$$

where $\boldsymbol{\epsilon} := \boldsymbol{v} - \mathbf{1}$.

Proof. The *linear manifold tangent* to $\mathcal{M}$ at a nominal operating point $\boldsymbol{x}_\bullet$ is given by

$$\mathcal{M}_\bullet := \left\{\boldsymbol{x} \in \mathbb{R}^{4n} : \boldsymbol{F}(\boldsymbol{x}_\bullet)\,(\boldsymbol{x} - \boldsymbol{x}_\bullet) = \mathbf{0}_{2n}\right\}, \qquad (20)$$

where

$$\boldsymbol{F}(\boldsymbol{x}_\bullet) = \begin{bmatrix} \frac{\partial \mathcal{F}}{\partial \boldsymbol{v}}(\boldsymbol{x}_\bullet) & \frac{\partial \mathcal{F}}{\partial \boldsymbol{\theta}}(\boldsymbol{x}_\bullet) & \frac{\partial \mathcal{F}}{\partial \boldsymbol{p}}(\boldsymbol{x}_\bullet) & \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}}(\boldsymbol{x}_\bullet) \end{bmatrix} \qquad (21a)$$

$$= \begin{bmatrix} \mathsf{Re}\,\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\bullet) & \mathsf{Re}\,\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\bullet) & -\mathbf{I}_n & \mathbf{0}_n \\ \mathsf{Im}\,\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\bullet) & \mathsf{Im}\,\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\bullet) & \mathbf{0}_n & -\mathbf{I}_n \end{bmatrix} \qquad (21b)$$

$$= \begin{bmatrix} \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\bullet) & \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\bullet) & -\mathbf{I}_n & \mathbf{0}_n \\ \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\bullet) & \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\bullet) & \mathbf{0}_n & -\mathbf{I}_n \end{bmatrix}. \qquad (21c)$$

Let $\boldsymbol{\omega} := \boldsymbol{\gamma} + \mathrm{j}\boldsymbol{\beta} \in \mathbb{C}^n$ denote the vector of self-admittances of each node. Then, following [13, 5.10], the Jacobian of complex power injections with respect to voltage phase angles is

$$\begin{aligned} \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\star) &= \mathrm{j}\,\mathsf{diag}(\boldsymbol{u}_\star)\left(\mathsf{diag}\left(\underline{\boldsymbol{Y}\boldsymbol{u}_\star}\right) - \underline{\boldsymbol{Y}}\,\mathsf{diag}\left(\underline{\boldsymbol{u}_\star}\right)\right) \\ &= \mathrm{j}\mathbf{I}_{n\times n}\left(\mathsf{diag}\left(\underline{\boldsymbol{Y}}\mathbf{1}_n\right) - \underline{\boldsymbol{Y}}\mathbf{I}_{n\times n}\right) \\ &= \mathrm{j}\left(\mathsf{diag}\left(\underline{\boldsymbol{\omega}}\right) - \underline{\boldsymbol{Y}}\right); \end{aligned}$$

similarly, with respect to the voltage magnitudes

$$\begin{aligned} \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\star) &= \mathsf{diag}(\boldsymbol{u})\left(\mathsf{diag}\left(\underline{\boldsymbol{Y}\boldsymbol{u}}\right) + \underline{\boldsymbol{Y}}\,\mathsf{diag}\left(\underline{\boldsymbol{u}}\right)\right)\mathsf{diag}(\boldsymbol{v})^{-1} \\ &= \mathbf{I}_{n\times n}\left(\mathsf{diag}\left(\underline{\boldsymbol{Y}}\mathbf{1}_n\right) + \underline{\boldsymbol{Y}}\mathbf{I}_{n\times n}\right)\mathbf{I}_{n\times n}^{-1} \\ &= \mathsf{diag}\left(\underline{\boldsymbol{\omega}}\right) + \underline{\boldsymbol{Y}}. \end{aligned}$$

Substituting the above into (21) yields the desired result.

Now, we show that the assumption that the network is a tree, or radial, ensures that the linear power flow matrix $\boldsymbol{M}$ can be inverted in the analytical form given in (18). Note that as $\boldsymbol{G} \succ 0$, both Schur complements of $\boldsymbol{M}$ exist.

Moreover, setting $\boldsymbol{S}$ to be the Schur Complement of $\boldsymbol{M}$ in $-\boldsymbol{G}$, we obtain that the inverse of $\boldsymbol{S}$ is, in fact, the *resistance matrix* $\boldsymbol{R}$, since

$$\begin{aligned} \boldsymbol{S}^{-1} &:= \left(\boldsymbol{G} + \boldsymbol{B}\boldsymbol{G}^{-1}\boldsymbol{B}\right)^{-1} \\ &= \left(\boldsymbol{A}^\top \mathsf{diag}(\boldsymbol{g})\boldsymbol{A} + \boldsymbol{A}^\top \mathsf{diag}(\boldsymbol{b})\,\mathsf{diag}(\boldsymbol{g})^{-1}\,\mathsf{diag}(\boldsymbol{b})\boldsymbol{A}\right)^{-1} \\ &= \left(\boldsymbol{A}^\top \mathsf{diag}\left(\left[\frac{g_{ij}^2 + b_{ij}^2}{g_{ij}}\right]_{ij\in\mathcal{E}}\right)\boldsymbol{A}\right)^{-1} \\ &= \boldsymbol{A}^{-1}\,\mathsf{diag}\left(\left[\frac{g_{ij}}{g_{ij}^2 + b_{ij}^2}\right]_{ij\in\mathcal{E}}\right)\boldsymbol{A}^{-\top} \\ &:= \boldsymbol{A}^{-1}\,\mathsf{diag}(\boldsymbol{r})\,\boldsymbol{A}^{-\top} \\ &:= \boldsymbol{R}. \end{aligned}$$

Further calculation reveals that the Schur complement of $\boldsymbol{M}$ in $\boldsymbol{G}$ is $-\boldsymbol{S}$. Therefore, applying well-known block matrix inversion identities yields

$$\boldsymbol{F}^{-1} = \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix}^{-1} \qquad (22a)$$

$$= \begin{bmatrix} \boldsymbol{S}^{-1} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & -\boldsymbol{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n\times n} & -\boldsymbol{B}\boldsymbol{G}^{-1} \\ \boldsymbol{B}\boldsymbol{G}^{-1} & \mathbf{I}_{n\times n} \end{bmatrix} \qquad (22b)$$

$$= \begin{bmatrix} \boldsymbol{S}^{-1} & -\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} \\ -\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} & -\boldsymbol{S}^{-1} \end{bmatrix}. \qquad (22c)$$

Finally, we must compute the off-diagonal matrices of (22). We obtain

$$\begin{aligned} -\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} &= -\boldsymbol{A}^{-1}\,\mathsf{diag}\left(\left[\frac{g_{ij}}{g_{ij}^2 + b_{ij}^2}\right]_{ij}\right)\mathsf{diag}(\boldsymbol{b} \oslash \boldsymbol{g})\boldsymbol{A}^{-\top} \\ &= \boldsymbol{A}^{-1}\,\mathsf{diag}\left(\left[\frac{-b_{ij}}{g_{ij}^2 + b_{ij}^2}\right]_{ij\in\mathcal{E}}\right)\boldsymbol{A}^{-\top} \\ &:= \boldsymbol{A}^{-1}\,\mathsf{diag}(\boldsymbol{x})\,\boldsymbol{A}^{-\top} \\ &:= \boldsymbol{X}, \end{aligned}$$

where $\oslash$ denotes element-wise division. Therefore, we have obtained that the inverse of the linear power flow matrix $\boldsymbol{M}$ is

$$\begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{S}^{-1} & -\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} \\ -\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} & -\boldsymbol{S}^{-1} \end{bmatrix} \qquad (23a)$$

$$= \begin{bmatrix} \boldsymbol{R} & \boldsymbol{X} \\ \boldsymbol{X} & -\boldsymbol{R} \end{bmatrix}, \qquad (23b)$$

as desired. $\square$